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BOOK REVIEWS

Elliptic pseudo-differential operators–an abstract theory, by H. O. Cordes, Lecture Notes in Math., vol. 756, Springer-Verlag, Berlin and New York, 1979, 331 pp., $18.00.

Pseudodifferential operators (often called ψDOs) are generalizations of differential operators, and they arose to treat problems in partial differential equations. One common characterization of a ψDO is as an operator of the form

$$Pu(x) = \int p(x, \xi)e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi \qquad (x, \xi \in \mathbf{R}^n) \tag{1}$$

where $\hat{u}(\xi) = (2\pi)^{-n}\int e^{-ix\cdot\xi}u(x)\, dx$ is the Fourier transform of u. Formula (1) defines a differential operator in case $p(x, \xi)$ is a polynomial in ξ. More generally, $p(x, \xi)$ can belong to a symbol class, such as the symbol class of Hörmander

$$p(x, \xi) \in S^m_{\rho,\delta} \Leftrightarrow \left|D^\beta_x D^\alpha_\xi p(x, \xi)\right| \leqslant C_{\alpha\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \tag{2}$$

or other classes, e.g., due to Beals and Fefferman **[1]**, **[2]**, or Hörmander **[11]**. Such operators captured the attention of many mathematicians, not necessarily primarily interested in partial differential equations, in the mid '60s, because of the role they played in the proof of the Atiyah-Singer index theorem, particularly in the production of families of operators known to be Fredholm, connecting together two elliptic differential operators with homotopic principal symbol, to prove the index of such an operator depends only on the homotopy class of its principal symbol.

An operator on a Hilbert space H is Fredholm if and only if it is invertible modulo the algebra $\mathcal{K}$ of compact operators. Thus, given a *-algebra $\mathfrak{A}_0$ of ψDOs, say of the form (1) with $p(x, \xi)$ perhaps belonging to a subclass of symbols of the form (2), to study Fredholm properties of elements of $\mathfrak{A}_0$ it is natural to look at the quotient algebra $\mathfrak{A}/\mathcal{K}$, where $\mathfrak{A}$ is the L^2-operator norm closure of $\mathfrak{A}_0$, perhaps with $\mathcal{K}$ thrown in. If $\mathfrak{A}$ acts irreducibly on H and contains one compact operator, as is often the case, it is not hard to show $\mathfrak{A}$ contains $\mathcal{K}$ (see **[15**, p. 192]). If $\mathfrak{A}_0$ consists of operators of the form (1), (2), with $m = 0$, $\rho = 1$, $\delta = 0$, and $p(x, \xi)$ well behaved at infinity, then commutators $[P, Q] = PQ - QP$ of elements of $\mathfrak{A}_0$ are compact and hence $\mathfrak{A}/\mathcal{K}$ is a commutative C^* algebra. The same holds if $\mathfrak{A}_0$ is the algebra of "classical" pseudodifferential operators of order zero on a compact manifold X, without boundary. Thus, $\mathfrak{A}/\mathcal{K}$ is isomorphic to $C(M)$, the algebra of continuous complex valued functions on a compact Hausdorff space M, which in the case of the last mentioned example turns out to be $S^*(X)$, the cosphere bundle of X. An element A of $\mathfrak{A}_0$ thus gives rise to a function $\sigma(A)$ on M, and A is Fredholm if and only if $\sigma(A)$ is nowhere vanishing on M. (To see this,