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STABLE AND L^2-COHOMOLOGY OF ARITHMETIC GROUPS

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Introduction. In [1], [2] we gave a range of dimensions in which the real cohomology of an arithmetic or S-arithmetic subgroup Γ of a connected semi-simple group G over $\mathbf{Q}$ is naturally isomorphic to the space of harmonic forms on the quotient $X = G(\mathbf{R})/K$ of the group $G(\mathbf{R})$ of real points of G by a maximal compact subgroup K which are invariant under Γ and the identity component $G(\mathbf{R})^0$ of $G(\mathbf{R})$, and indicated some applications to the stable cohomology of classical arithmetic groups and to algebraic K-theory. In this note we first state an extension to nontrivial coefficients, since this has become of interest in topology and K-theory [7]. A chief tool in [2] was the proof that $H^*(\Gamma; \mathbf{C})$ could be computed using differential forms on $\Gamma\backslash X$ which have "logarithmic growth" at infinity. Theorem 2 extends this to more general growth conditions. This can be used to show that certain L^2-harmonic forms are not cohomologous to zero [9]. In §§3, 4, 5 we consider the L^2-cohomology space $H_{(2)}(\Gamma\backslash X)$ and relate it to the spectral decomposition of the space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$. Theorem 4 gives a sufficient condition under which it is finite dimensional, hence isomorphic to the space of square integrable harmonic forms, and §5 a series of examples in which it is not. For convenience, we assume G simple over $\mathbf{Q}$ and Γ torsion-free.

1. Let P_0 be a minimal parabolic $\mathbf{Q}$-subgroup of G, S a maximal $\mathbf{Q}$-split torus of P_0, N the unipotent radical of P and $\mathfrak{n}$ the Lie algebra of N. Let $X(S)$ be the group of rational characters of S and $\rho \in X(S)$ be such that $a^{2\rho} = \det \operatorname{Ad} a|_{\mathfrak{n}}$ for $a \in S$. For $\mu \in X(S)$ let $c(G, \mu)$ be the maximum of q such that $\rho - \mu - \eta > 0$, where η runs through the weights of S in $\Lambda^q\mathfrak{n}$. Let $c(G) = c(G, 0)$. If (r, E) is a finite-dimensional complex representation of $G(\mathbf{C})$, we let $c(G, r)$ be the minimum of $c(G, \mu)$, where μ runs through the weights of r with respect to S. It is easily seen that $c(G) \geqslant \Sigma_i c(G_i)$, where G_i runs through the simple factors of $G(\mathbf{C})$, and $c(G_i)$-is defined similarly, and that $c(G_i)$ is equal to $[(l-1)/2], l-1, l-2, l-1, 7, 13, 25, 5, 1$ if G_i is of type $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$.

THEOREM 1. *The natural homomorphism $H^*(\mathfrak{g}, \mathfrak{k}; E)^\Gamma \longrightarrow H^q(\Gamma; E)$ is injective for $q \leqslant c(G, r)$, surjective if in addition $q < \operatorname{rk}_{\mathbf{R}} G$. If $E^G = (0)$, then $H^q(\Gamma; E) = 0$ for $q \leqslant c(G, r)$, $(\operatorname{rk}_{\mathbf{R}} G - 1)$. If G is simply connected, these*

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