BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 2, September 1980

VARIETIES AND UNIVERSAL MODELS IN THE THEORY OF COMBINATORIAL GEOMETRIES

BY JEFF KAHN AND JOSEPH P. S. KUNG

One of the most attractive but little studied ideas in the theory of combinatorial geometries (or matroids) [2], [4] is the notion of a hereditary class of geometries. A *hereditary class* of (finite) geometries is a collection of geometries which is closed under taking minors and direct sums. Thus, hereditary classes are direct analogues of varieties in universal algebra. Although varieties are highly structured collections of objects—free objects exist, for example—hereditary classes are relatively unstructured. In order to obtain reasonable results, it is necessary to impose some regularity conditions. One possibility is to postulate the existence of free (or rather, cofree) objects. A *sequence of universal models* for a hereditary class T of geometries is a sequence (T_n) of geometries in T with rank $(T_n) = n$ satisfying the universal property: if G is a geometry in T of rank n, then G is a subgeometry of T_n. A *variety* of geometries is a hereditary class with a sequence of universal models. Rather surprisingly, it is possible to classify varieties of geometries.

Before stating our result, we need to describe two simple sequences of geometries. Let M_1 be the rank one geometry and M_2 be the line with $q + 1$ points. Let $M_{2n}(q) = M_2 \oplus \cdots \oplus M_2$ and $M_{2n+1}(q) = M_2 \oplus \cdots \oplus M_2 \oplus M_1$. The subgeometries of these geometries form a variety called the *variety of matchstick geometries of order q*. Now, let B_n be the Boolean algebra on the set $\{1, \ldots, n\}$. On each of the lines $\overline{12}, \overline{23}, \ldots, \overline{(n-1)n}$, add $q - 1$ points in general position and call the resulting geometry $O_n(q)$. The subgeometries of these geometries also form a variety called the *variety of origami geometries of order q*.

THEOREM. *Let T be a variety of geometries. Then, T is one of the following collections*:

1. *the variety of free geometries*;
2. *the variety of matchstick geometries of order q*;
3. *the variety of origami geometries of order q*;
4. *the variety of all geometries coordinatizable over a fixed finite field $GF(q)$*;
5. *the variety of voltage-graphic geometries with voltages in a fixed finite group A.*

Received by the editors March 5, 1980.
AMS (MOS) subject classifications (1970). Primary 05B35, 08A15.

0002-9904/80/0000-0405/$01.50