BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 2, March 1980

THE CELL-LIKE MAPPING PROBLEM

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A major unresolved issue in topology is whether or not there is a dimension raising cell-like mapping. A *cell-like map* $f: X \longrightarrow Y$ is a proper mapping between metrizable spaces such that $f^{-1}(y)$ has the shape of a point for each $y \in Y$ (i.e., every map of $f^{-1}(y)$ to a polyhedron is null-homotopic or, equivalently for finite dimensional X, $f^{-1}(y)$ admits a cellular embedding in a Euclidean space).

During the period of his pioneering work on cellular decompositions of 3-space, Bing asked if the quotient space of such a decomposition is an ANR **[Bi]**. In the mid-sixties, Armentrout asked if the quotient space of a cellular decomposition of E^3 is finite dimensional **[Ar]**. It was shown in **[Koz-1]** that these two questions are equivalent and both are settled affirmatively by the following theorem.

THEOREM. *If $f: X \longrightarrow Y$ is a cell-like mapping defined on a subset of a 3-manifold, then* $\dim Y \leqslant 3$.

The theorem is best understood by considering the case with X a compact subset of E^3. The Sphere Theorem easily implies that (1) a cell-like subset of E^3 has arbitrarily small aspherical open neighborhoods and (2) each component of the intersection of two aspherical open subsets of E^3 is aspherical. In **[Koz-2]**, the cell-like mapping problem is reduced to an extension problem for maps into polyhedra; this extension problem is solvable for maps into aspherical polyhedra. These facts are combined to prove the theorem by producing an ϵ-map of Y into E^3 for each $\epsilon > 0$.

It was known previously that cell-like maps on 1-dimensional spaces do not raise dimension. This is proved by using the Vietoris Mapping Theorem to conclude that the image has (integral) cohomological dimension at most 1 and then appealing to the fact that covering dimension agrees with cohomological dimension in this case (both are "classified" by extensions of maps into S^1). The precise connection between the cell-like mapping problem and the classical question of whether or not covering dimension and cohomological dimension agree for compacta was recently established by R. D. Edwards **[Ed-1]**; he announced that a compactum with cohomological dimension $\leqslant n$ is the cell-like image of a compactum with covering dimension $\leqslant n$.

It seems appropriate to sketch two reductions of the cell-like mapping problem for such maps on ANR's and manifolds. First, the result of Bothe **[Bo]**

Received by the editors June 1, 1979.

AMS (MOS) subject classifications (1970). Primary 54F45, 57A10; Secondary 54C55, 54A35.

Key words and phrases. Cell-like, aspherical, 3-manifolds.

[1]Research supported in part by an NSF grant.