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Differential forms in mathematical physics, by C. von Westenholz, Studies in Mathematics and its Applications 3, North-Holland Publishing Company, Amsterdam, 1978, xv + 487 pp., $62.25.

An example or two will give the flavor of the subject. First, let M be an n-dimensional smooth differentiable manifold, thought of as the configuration space of a mechanical system with n degrees of freedom. Each point of M has a neighborhood with a local coordinate system $(q^1, \ldots, q^n)$. When the system is in motion we need not only the coordinates q^i of a point of M, but also the momentum vector $(p_1, \ldots, p_n)$ at q. Thus we are lead to the *phase space*, or *cotangent bundle*, of M, denoted T^*M. This space already has an interesting structure: the differential form of degree one with local expression

$$\omega = \sum p_i dq^i$$

is really a global quantity on T^*M. Its exterior derivative

$$\Omega = d\omega = \sum dp_i \wedge dq^i$$

is automatically a global quantity on T^*M, an exterior differential form of degree two (skew-symmetric covariant 2-tensor). The equations of motion of the system are described in the following way by a real function H on M, called the *Hamiltonian* of the motion:

There is a contraction process, called the *interior product*, that contracts Ω with any vector field X on T^*M to produce a differential form of degree one (one-form for short) $X \lrcorner \Omega$. If Ω is thought of as an alternating bilinear functional on vector fields, then

$$(X \lrcorner \Omega)(Y) = \Omega(X, Y)$$

exhibits $X \lrcorner \Omega$ as a linear functional on vector fields, that is, a one-form. It turns out that $X \to X \lrcorner \Omega$ is an isomorphism on the space of vector fields onto the space of one-forms, so there is a unique vector field X_H such that

$$X_H \lrcorner \Omega = dH.$$

A short calculation in the local coordinate system q^i, p_i yields

$$X_H = \sum \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}.$$

(A vector field here is thought of as a directional derivation on the space of real functions.) Therefore, in local coordinates, a curve $q^i = q^i(t), p_i = p_i(t)$ is a trajectory of X_H, provided that

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

These are precisely Hamilton's equations of motion of the system.

From this example we see that a lot of the structure of differential geometry: manifolds, bundles over manifolds, induced structures, vector