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COLLOQUIUM LECTURES ON GEOMETRIC MEASURE THEORY[1]

BY HERBERT FEDERER

1. Introduction. The early discoveries of measure theory, at the start of this century, led to a very good understanding of how subsets of Euclidean n space

$$\mathbf{R}^n$$

behave with respect to n dimensional Lebesgue measure

$$\mathcal{L}^n.$$

Much of the theory of functions was revolutionized by Lebesgue's method of integration. This paved the way for great advances in Fourier analysis. Furthermore Lebesgue's contributions to measure theory made possible the application of direct methods in the one dimensional calculus of variations, which soon developed to a highly satisfactory stage.

Many two dimensional variational problems, including some versions of the problem of Plateau, have also been solved through ingenious extension of Lebesgue's methods, making use of conformal parametrizations and Dirichlet integrals. However, these classical methods have failed to give significant results for the part of the calculus of variations involving parametric integrals over m dimensional surfaces in case m exceeds two. Concrete examples show that sets representing solutions of high dimensional variational problems can have very complicated singularities. To account for geometric actuality, and in order to prove general existence theorems, one must abandon the idea of describing all the competing surfaces by continuous maps from a single predetermined parameter space. One should rather think of surfaces as m dimensional mass distributions, with tangent m vectors attached. Then the boundary conditions must be expressed in a manner quite different from functional restriction–for instance through a boundary operator as in algebraic topology.

It took five decades, beginning with Carathéodory's fundamental paper on measure theory in 1914, to develop the intuitive conception of an m dimensional surface as a mass distribution into an efficient instrument of mathematical analysis, capable of significant applications in the calculus of variations. The first three decades were spent learning basic facts on how subsets of $\mathbf{R}^n$ behave with respect to m dimensional Hausdorff measure

$$\mathcal{H}^m.$$

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