BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1978

ON THE THEORY OF Π^1_3 SETS OF REALS

BY A. S. KECHRIS[1] AND D. A. MARTIN

Communicated by Solomon Feferman, June 2, 1977

1. An ordinal basis theorem. Assuming that $\forall x \in \omega^\omega$ ($x^\#$ exists), let u_α be the αth uniform indiscernible (see [3] or [2]). A canonical coding system for ordinals $< u_\omega$ can be defined by letting $WO_\omega = \{w \in \omega^\omega : w = \langle n, x^\#\rangle$, for some $n \in \omega, x \in \omega^\omega\}$ and for $w = \langle n, x^\#\rangle \in WO_\omega$, $|w| = \tau_n^{L[x]}(u_1, \ldots, u_{k_n})$, where τ_n is the nth term in a recursive enumeration of all terms in the language of $ZF + V = L[\dot{x}]$, $\dot{x}$ a constant, taking always ordinal values. Call a relation $P(\xi, x)$, where ξ varies over u_ω and x over ω^ω, Π^1_k if $P^*(w, x) \Leftrightarrow w \in WO_\omega \wedge P(|w|, x)$ is Π^1_k. An ordinal $\xi < u_\omega$ is called Δ^1_k if it has a Δ^1_k notation i.e. $\exists w \in WO_\omega$ ($w \in \Delta^1_k \wedge |w| = \xi$).

THEOREM 1 ($ZF + DC +$ DETERMINACY (Δ^1_2)). *Every nonempty Π^1_3 subset of u_ω contains a Δ^1_3* ordinal.

COROLLARY 2 ($ZF + DC +$ DETERMINACY (Δ^1_2)). *Π^1_3 is closed under quantification over ordinals $< u_\omega$ i.e. if $P(\xi, x)$ is Π^1_3 so are $\exists\xi P(\xi, x)$, $\forall\xi P(\xi, x)$.*

COROLLARY 3 ($ZF + DC + AD$). *The class of Π^1_3 sets of reals is closed under $< \delta^1_3$ intersections and unions.*

Martin [3] has proved the corresponding result for Δ^1_3.

2. A Kleene theory for Π^1_3. Kleene has characterized the Π^1_1 relations as those which are inductive (see [7]) on the structure $\langle \omega, < \rangle = \mathcal{Q}_1$. Let $j_m: u_\omega \longrightarrow u_\omega$, $m \geq 1$, be defined by letting

$$j_m(u_i) = \begin{cases} u_i, & \text{if } i < m, \\ u_{i+1}, & \text{if } i \geq m, \end{cases}$$

and then

$$j_m(\tau_n^{L[x]}(u_1, \cdots u_{k_n})) = \tau_n^{L[x]}(j_m(u_1) \cdots j_m(u_{k_n})).$$

Let R be the relation on u_ω coding these embeddings, i.e.

$$R = \{(m, \alpha, \beta): m \in \omega \wedge \alpha, \beta < u_\omega \wedge j_m(\alpha) = \beta\}.$$

Put $\mathcal{Q}_3 = \langle u_\omega, <, R\rangle$.

AMS (MOS) subject classifications (1970). Primary 04A15, 02K30, 28A05, 54H05; Secondary 02F35, 02K25, 02K35, 04A30.

[1] Research partially supported by NSF Grant MCS 76-17254.