BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1978

NATURAL STRUCTURES ON SEMIGROUPS WITH INVOLUTION

BY MICHAEL P. DRAZIN

Communicated by Olga Taussky Todd, January 21, 1977

Let S be any semigroup, i.e. a set furnished with an associative binary operation, denoted by juxtaposition. An *involution* on S will mean a bijection $u \longrightarrow u^*$ of S onto itself, satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$. Such an involution is called *proper* (cf. [3, p. 74]) iff $a^*a = a^*b = b^*a = b^*b$ implies $a = b$. A *proper $*$-semigroup* will mean a pair $(S, *)$ where S is a semigroup and $*$ is a specified proper involution of S; in practice, we write S as an abbreviation for $(S, *)$. From now on, S will denote an arbitrary proper $*$-semigroup.

Obvious natural special cases are

(i) all proper $*$-rings, with "properness" (Herstein [4, p. 794] prefers to say "positive definiteness") as customarily defined (cf. [5, p. 31], [1, p. 10]) via $u^*u = 0$ implying $u = 0$ (in particular, with the obvious choices for $*$, all commutative rings with no nonzero nilpotent elements, all Boolean rings, the ring $\mathcal{B}(H)$ of all bounded linear operators on any complex Hilbert space H, and the ring $M_n(\mathrm{C})$ of all $n \times n$ complex matrices), and, only slightly less trivially,

(ii) all inverse semigroups (in particular, all groups).

We make a start here towards showing that the proper $*$-semigroup axioms allow the simultaneous development of a surprisingly rich common theory of these special cases. While there is clearly little likelihood of learning anything new about groups or Boolean rings by such an approach, none of the results about proper $*$-semigroups which we state below has previously been noted even for $n \times n$ matrices (still less for $\mathcal{B}(H)$ or for $*$-rings), and most provide new information also about inverse semigroups.

We recall that an element $a \in S$ is called *regular* iff $a \in aSa$, and *$*$-regular* iff there is an $x \in S$ with

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

For given $a \in S$, there can (cf. [6, Theorem 1]) be at most one such x, and, if any x exists, we write $x = a^\dagger$ (known as the *Moore-Penrose generalized inverse* of a). It is known (cf. [7]) that a is $*$-regular iff aa^* and a^*a are both regular; let $V_*(S)$ denote the set of all such a. (For example, $V_*(\mathcal{B}(H))$ consists [3, p. 73]

AMS (MOS) subject classifications (1970). Primary 06A10, 15A09, 16A28, 20M99, 46K99; Secondary 16A30.