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Le mouvement brownien relativiste, by Jean-Pierre Caubet, Lecture Notes in Mathematics, no. 559, Springer-Verlag, Berlin, Heidelberg, New York, 1976, ix + 212 pp., $10.20.

It is just 25 years since Imre Fényes [2] discovered the Markov process associated with a solution of the Schrödinger equation. This process is easy to describe. Suppose that we have a solution ψ of the Schrödinger equation

$$\frac{\partial}{\partial t}\psi(x, t) = i\left(\frac{1}{2}\Delta - V\right)\psi(x, t). \tag{1}$$

Here x ranges over $\mathbf{R}^n$ and t over $\mathbf{R}$ and for each t the function $\psi_t(x) = \psi(x, t)$ is in $L^2(\mathbf{R}^n)$. We have set $\hbar = m = 1$, and V is the operator of multiplication by a possibly time-dependent real function (also denoted by V). The quantity $\|\psi_t\|_2^2$ is independent of t and may be normalized to 1, so that $\rho_t = |\psi_t|^2$ is the density of a probability measure on $\mathbf{R}^n$. We may write $\psi = \exp(R + iS)$ and set $u = \operatorname{grad} R$, $v = \operatorname{grad} S$, $b = u + v$. Then the diffusion process with drift b–that is, the Markov process satisfying

$$dx(t) = b(x(t), t)\,dt + dw(t) \tag{2}$$

where w is the Wiener process–and with initial distribution ρ_0 has the probability distribution ρ_t at all times t. Conversely, given such a diffusion process (2) we may let b_* be the backward drift and set $u = (b - b_*)/2$, $v = (b + b_*)/2$. Then u is the gradient of $\frac{1}{2}\log\rho$. If we assume that v is also a gradient then there is a unique V and a unique solution ψ of (1) such that with $\psi = \exp(R + iS)$ we have $u = \operatorname{grad} R$ and $v = \operatorname{grad} S$.

To get a better idea of these processes let us consider a few examples.

Consider the Wiener process itself where as customary we require the particle to be at the origin at time 0. Then the drift $b(x, t)$ is 0 for $t > 0$, but for $t < 0$ the particle is destined to go from x to the origin in $|t|$ units of time and the drift is $b(x, t) = x/t$. Similarly $b_*(x, t) = 0$ for $t < 0$ but $b_*(x, t) = x/t$ for $t > 0$. Now it is straightforward to compute u, v, R, S, ψ, and V. We find

$$\psi = (\pi t)^{-n/4}\exp\left(-\frac{1}{2}\frac{x^2}{|t|} + i\frac{1}{2}\frac{x^2}{t}\right),$$

which satisfies (1) with $V = x^2/4t^2$. The graph of this time-dependent potential is a paraboloid which is very shallow for large negative t but which snaps shut as $t \to 0$ and then opens out again, forcing the particle to go through the origin at $t = 0$. Thus this process is not free; it is subject to a force $F = -\operatorname{grad} V = -x/2t^2$.

A shortcut for computing V is available from the fact [3], [4] that Newton's law $F = ma$ holds, where a is the mean acceleration defined as follows. For a stochastic process x we define