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HILBERT'S TWELFTH PROBLEM AND L-SERIES

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Let k be a totally real number field of degree $n \geqslant 2$ with conjugate fields $k = k^{(1)}, \ldots, k^{(n)}$. Let $I(\mathfrak{f})$ denote the group of fractional ideals of k generated by those integral ideals relatively prime to a given integral ideal, $\mathfrak{f}$. Let $S(\mathfrak{f})$ denote the subgroup of $I(\mathfrak{f})$ generated by those principal integral ideals (α) with $\alpha \equiv 1 \pmod{\mathfrak{f}}$. The quotient group $H = I(\mathfrak{f})/S(\mathfrak{f})$ is the ray class group $(\mathrm{mod}\ \mathfrak{f})$ of k and corresponds via class field theory to a totally real abelian extension F of k.

We define the character of sign $\lambda(\alpha)$ on k by

$$\lambda(\alpha) = \prod_{j=2}^{n} \operatorname{sgn}(\alpha^{(j)}).$$

Let $\mathfrak{C}_0$ denote the subgroup of all (α) in $S(\mathfrak{f})$ such that $\lambda(\alpha) = 1$ and $\mathfrak{J}$ the set of all (α) in $S(\mathfrak{f})$ such that $\lambda(\alpha) = -1$. It can happen that $\mathfrak{C}_0 = \mathfrak{J} = S(\mathfrak{f})$. The condition that this not occur is that for all units ϵ of k congruent to 1 $(\mathrm{mod}\ \mathfrak{f})$, we must have $\lambda(\epsilon) = 1$. We assume that $\mathfrak{f}$ satisfies this condition, and let $G = I(\mathfrak{f})/\mathfrak{C}_0$. By class field theory, G corresponds to a real abelian extension K of k which is a quadratic extension of F.

For any $\mathfrak{C}$ in G, let

$$\zeta(s, \mathfrak{C}) = \sum_{\mathfrak{A} \in \mathfrak{C}} N(\mathfrak{A})^{-s}$$

where the sum is over all integral ideals $\mathfrak{A}$ of $\mathfrak{C}$. Let

$$\epsilon(\mathfrak{C}) = \exp\,[2\zeta'(0, \mathfrak{C})], \qquad \epsilon = \epsilon(\mathfrak{C}_0).$$

CONJECTURE 1. *The numbers $\epsilon(\mathfrak{C})$ are conjugate algebraic integers in K. If $\mathfrak{p}$ is a first degree prime ideal in $\mathfrak{C}$ of norm p then the explicit reciprocity law of class field theory is given by*

$$\epsilon^p \equiv \epsilon(\mathfrak{C}) \pmod{\mathfrak{p}}.$$

Our conjecture thus provides an answer to Hilbert's twelfth problem for totally real fields k. The purpose of this note is to present the first numerical example of Conjecture 1 with a nonabelian ground field k. Conjecture 1 implies that $\epsilon(\mathfrak{C}\mathfrak{J}) = \epsilon(\mathfrak{C})^{-1}$ is a unit, that

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