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A GOOD ALGORITHM FOR LEXICOGRAPHICALLY OPTIMAL FLOWS IN MULTI-TERMINAL NETWORKS

BY NIMROD MEGIDDO

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ABSTRACT. Let a network have several sources and sinks. For any flow f let σ^f and τ^f denote the vectors of net flows out of the sources and into the sinks, respectively, arranged in order of increasing magnitude. Our algorithm computes an f for which both σ^f and τ^f are lexicographic maxima. For a network with n nodes this algorithm terminates within $O(n^5)$ operations.

1. The problem. A *network* (N, c) consists of a set of *nodes* $N = \{1, \ldots, n\}$ $(n \geqslant 1)$ and an $n \times n$ nonnegative matrix c of *capacities*. $S \subset N$ is a nonempty set of *sources* and $T \subset N$ $(T \cap S = \emptyset)$ is the set of *sinks*. A flow f is an $n \times n$ matrix such that $0 \leqslant f_{ij} \leqslant c_{ij}$ $(i, j \in N)$ and $\Sigma_{j=1}^n (f_{ij} - f_{ji}) = 0$ for $i \notin S \cup T$. Denote $s = |S|$, $t = |T|$.

Let σ^f $[\tau^f]$ denote the s-tuple [t-tuple] of the numbers $\Sigma_{j=1}^n (f_{ij} - f_{ji})$, $i \in S$ $[\Sigma_{j=1}^n (f_{ji} - f_{ij}), i \in T]$ arranged in order of increasing magnitude. f is called *optimal* if it maximizes both σ^f and τ^f in the lexicographic orders on R^s and R^t, respectively.

Optimal flows reduce to maximum flows (see [5]) when $s = t = 1$. Existence of optimal flows is proved in [7]. The goal of this note is to present a good algorithm (in the sense of [2]) for finding an optimal flow.

2. The algorithm. The algorithm has two phases. In Phase I the network is decomposed to two networks, one with a single source and t sinks, and the other with s sources and a single sink. In Phase II optimal flows are found in these two networks.

PHASE I. Find a flow f which maximizes $\Sigma_{i \in S} \Sigma_{j=1}^n (f_{ij} - f_{ji})$. Any of the following algorithms may be utilized: Karzanov [6] terminates within $O(n^3)$ operations, Dinic [1] and Even and Tarjan [4] $O(n^4)$, and Edmonds and Karp [3] $O(n^5)$. During the computation of f a set X, $S \subset X \subset N \backslash T$, is generated such that for $i \in X$ and $j \notin X$, $f_{ij} = c_{ij}$ and $f_{ji} = 0$. Next, construct the X-condensed and the $(N \backslash X)$-condensed networks (see [7]).

PHASE II. Find optimal flows in the X-condensed and the $(N \backslash X)$-condensed networks independently. These two are treated symmetrically and, hence, without loss of generality assume that $S = \{1\}$.

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