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A CHARACTERIZATION OF HARMONIC IMMERSIONS OF SURFACES

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Let S be an oriented surface with Riemannian metric ds^2, and M^n a Riemannian manifold of dimension $n \geqslant 2$. We present here a characterization of harmonic immersions $f: S \longrightarrow M^n$ which sheds some light on their differential geometric properties. While C^∞ smoothness is assumed throughout, less is needed.

To work on the Riemann surface determined by ds^2 on S, use conformal parameters $z = x_1 + ix_2$ which correspond to ds^2-isothermal coordinates x_1, x_2 on S. Given any local coordinates on M^n, write $f = (f^\alpha)$ and $f_i^\alpha = \partial f^\alpha / \partial x_i$ where $i = 1, 2$ and $\alpha, \beta, \gamma = 1, 2, \ldots, n$. An immersion $f: S \longrightarrow M^n$ is harmonic if and only if for each α and for any ds^2-isothermal coordinates x_1, x_2 on S

$$\partial^2 f^\alpha / \partial x_i^2 + \Gamma^\alpha_{\beta\gamma} f_i^\beta f_i^\gamma = 0,$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for the metric on M^n, and one sums on the indices β, γ and i.

To any real quadratic form $X = l_{ij} dx_i dx_j$ on S, associate on R the quadratic differential $\Omega(X, R)$ and the conformal metric $\Gamma(X, R)$ given by $4\Omega(X, R) = (l_{11} - l_{22} - 2il_{12})dz^2$ and $2\Gamma(X, R) = (l_{11} + l_{22}) dz d\bar{z}$ respectively. Thus $X = 2 \operatorname{Re} \Omega + \Gamma$ on R. (See [10].) Call $\Omega(X, R)$ holomorphic if and only if the coefficient of dz^2 is complex analytic in z for every conformal parameter z on R. An immersion $f: S \longrightarrow M^n$ yields many quadratic forms of interest, among them the induced metric I, and the second fundamental forms $\mathrm{II}(N)$ determined by choices of a unit normal vector field N.

DEFINITION. An immersion $f: S \longrightarrow M^n$ is *R-minimal* if and only if $\Omega(\mathrm{I}, R)$ is holomorphic, and $\Gamma(\mathrm{II}(N), R) \equiv 0$ for any choice (local or global) of a unit normal vector field N.

An R-minimal immersion is *minimal* if and only if R is the Riemann surface R_I determined on S by I. It is known that a conformal immersion $f: S \longrightarrow M^n$ is harmonic if and only if it is minimal. Indeed, this is established in [2] independent of the dimensions of S and M^n. By analogy, we have the following

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