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SECOND ORDER ELLIPTIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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We consider the mixed boundary value problem (MBVP) $Au = f$ in Ω, $B^+u = g^+$ in Γ^+, $u = g^-$ in Γ^- where Ω is a bounded open subset of R^n whose boundary Γ is divided into disjoint open subsets Γ^+ and Γ^- by an $(n-2)$-dimensional manifold ω in Γ. We assume $A = \Sigma_{|\alpha|\leq 2} a_\alpha(x)D^\alpha$ is a *properly elliptic* operator on $\overline{\Omega}$ and $B^+ = \Sigma_{j=1}^n b_j^+(x)D_j + b_0(x)$ is a *normal* boundary operator satisfying the *complementing condition* with respect to A on $\overline{\Gamma^+}$. The coefficients of the operators and Γ^+, Γ^- and ω are all assumed arbitrarily smooth.

Throughout, s will denote a real number with $s \not\equiv \frac{1}{2} \pmod 1$. For $G = R^n$, $R^n_\pm$, Ω or Γ, the Sobolev spaces $H^s(G)$ are as in Lions-Magenes [1]. Also $H^s(\Gamma^\pm)$ is the space of restrictions to $\Gamma^\pm$ of distributions in $H^s(\Gamma)$, with the infimum norm, and $H^s_A(\Omega) = \{u \in H^s(\Omega): Au \in L^2(\Omega)\}$ with the graph norm. Let γ_0: $H^s_A(\Omega) \to H^{s-1/2}(\Gamma)$ be the trace map, $r^\pm$: $H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma^\pm)$ the restriction maps, and $\gamma^- = r^-\gamma_0$. Then $B^+ = r^+B$ for some first-order normal boundary operator B on the whole of Γ.

Consider the maps $(A, \gamma^-, B^+)_s$ defined as

$$(A, \gamma^-, B^+): H^s(\Omega) \to H^{s-2}(\Omega) \times H^{s-1/2}(\Gamma^-) \times H^{s-3/2}(\Gamma^+) \quad \text{if } s > 3/2,$$

$$(A, \gamma^-, B^+): H^s_A(\Omega) \to L^2(\Omega) \times H^{s-1/2}(\Gamma^-) \times H^{s-3/2}(\Gamma^+) \quad \text{if } s < 3/2.$$

These maps are bounded for all s, by the condition of normality for $s < 3/2$ (see for example [1, §2.8.1]). The MBVP is called *well-posed* if there exists $s \not\equiv \frac{1}{2} \pmod 1$ for which $(A, \gamma^-, B^+)_s$ is Fredholm. A bounded linear operator between Hilbert spaces is called *α-semi-Fredholm* (αsF) if it has finite dimensional kernel and closed range, *β-semi-Fredholm* (βsF) if it has closed range with finite codimension, and *Fredholm* if it is αsF and βsF.

THEOREM. *For each $x \in \omega$ there is an open subset I_x of the reals such that for $s \not\equiv \frac{1}{2} \pmod 1$, $(A, \gamma^-, B^+)_s$ is Fredholm if and only if $s \in I = \bigcap_{x\in\omega} I_x$. Moreover, I is open and so the MBVP is well-posed if and only if I is nonempty. In fact, for each $x \in \omega$ there is a real number e_x determined algebraically*

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