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# TWO-POINT PADÉ TABLES AND $T$-FRACTIONS[1]

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Continued fractions of the form

$$1 + d_0 z + \frac{z}{1 + d_1 z} + \frac{z}{1 + d_2 z} + \cdots, \qquad d_n \in \mathbf{C}, \tag{1}$$

called $T$-fractions, were introduced by one of the authors in 1948 [7]. He showed that *corresponding* to a given formal power series (fps)

$$L = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{2}$$

at $z = 0$, there exists a unique $T$-fraction (1) with the property that the Taylor expansion about $z = 0$ of the $n$th approximant of (1) agrees with (2) up through the term $c_n z^n$. He further showed that every $T$-fraction *corresponds* to some fps (2) in the above sense. However, it is known that if all $d_n \neq 0$, then the approximants of the $T$-fraction expansion of $L$ are not in the Padé table of $L$.

Perron [5, p. 179] was the first to observe that every $T$-fraction (1), with all $d_n \neq 0$ for $n \geqslant 0$, *corresponds* to some formal Laurent series (fLs)

$$L^* = c_{-1}^* z + c_0^* + \frac{c_1^*}{z} + \frac{c_2^*}{z^2} + \cdots, \qquad c_{-1}^* \neq 0, \tag{3}$$

at $z = \infty$ in the sense that the Laurent expansion of the $n$th approximant of (1) agrees with (3) up through the term $c_{n-1}^*/z^{n-1}$ (see also [4], [8]). Attempts to relate the continued fraction expansion (1) at $\infty$ to that at 0 have been unsuccessful until now.

The concept of the Padé table [3] has recently been generalized to give rational approximants for formal Newton series called Newton-Padé approximants (see, for example, [2]) and for approximation alternately at 0 and $\infty$ called two-point Padé approximants (see, for example, [1], [6]). In this paper we show that the approximants of the $T$-fractions with all $d_n \neq 0$ for $n \geqslant 0$ are the $(n + 1, n)$ entries in the two-point Padé table of the series (2) and (3) to which the $T$-fraction corresponds. In addition we are able to give an explicit formula for the $d_n$ in terms of the $c_n$ and $c_m^*$ (see Equation (6)) and also an explicit formula for the

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