BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, March 1977

CLASS NUMBERS OF TOTALLY POSITIVE BINARY FORMS OVER TOTALLY REAL NUMBER FIELDS[1]

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Communicated by Olga Taussky Todd, October 30, 1976

Let (V, q) be a totally positive binary quadratic space over a totally real number field k. A lattice in V is a finitely generated $\mathfrak{o}_k$-submodule of V of rank 2, $\mathfrak{o}_k$ being the ring of integers in k. We can define the notions of class and genus on the set of all lattices in V (cf. [1]). The purpose of this note is to announce an explicit formula for the number of proper classes in the genus of any free lattice in V. The details will be published elsewhere.

1. A class number relation. Scaling q by a constant factor if necessary, we may assume that q represents 1. Then, the binary quadratic space (V, q) is k-isomorphic to $(k(\sqrt{-\delta}), N)$, δ being the discriminant of (V, q) and N the norm of $k(\sqrt{-\delta})$ to k. Let G be the kernel of the norm map ν: $R_{K/k}(\mathbf{G}_m) \longrightarrow \mathbf{G}_m$, where $K = k(\sqrt{-\delta})$, $R_{K/k}$ is the Weil functor of restricting the field of definition from K to k (cf. [6]) and $\mathbf{G}_m$ denotes the multiplicative group of non-zero elements in a universal domain containing k. Then, the algebraic torus G is nothing but the special orthogonal group of N, and the class number H of G over k, which is intrinsically defined, can be interpreted as the number of proper classes in the genus of any free lattice in K.

Consider the isogeny λ: $R_{K/k}(\mathbf{G}_m) \longrightarrow G \times \mathbf{G}_m$ defined by

$$\lambda(x) = (x^2\nu(x)^{-1}, \nu(x)).$$

If we identify the character groups of $R_{K/k}(\mathbf{G}_m)$, G, $\mathbf{G}_m$ by $\mathbf{Z}[\mathfrak{G}]$, $\mathbf{Z}[\mathfrak{G}]/\mathbf{Z}s$, $\mathbf{Z}$, respectively, $\mathfrak{G}$ being the Galois group of K/k and $s = \Sigma_{\sigma\in\mathfrak{G}}\, \sigma$, then the dual $\hat{\lambda}$: $\widehat{G \times \mathbf{G}_m} \longrightarrow \widehat{R_{K/k}(\mathbf{G}_m)}$ of λ is given by

$$\hat{\lambda}(\gamma \bmod \mathbf{Z}s, z) = zs + (2\gamma - S(\gamma)s),$$

where $S(\gamma) = \Sigma_{\sigma\in\mathfrak{G}}\, z_\sigma$ if $\gamma = \Sigma_{\sigma\in\mathfrak{G}}\, z_\sigma \cdot \sigma \in \widehat{R_{K/k}(\mathbf{G}_m)} = Z[\mathfrak{G}]$. The maps λ and $\hat{\lambda}$ induce naturally the following maps: λ_v: $R_{K/k}(\mathbf{G}_m)_v \longrightarrow (G \times \mathbf{G}_m)_v$ for each (finite or infinite) prime v of k, $\lambda_{\mathfrak{p}}^c : R_{K/k}(\mathbf{G}_m)_{\mathfrak{p}}^c \longrightarrow (G \times \mathbf{G}_m)_{\mathfrak{p}}^c$ for each finite prime $\mathfrak{p}$ of k, λ_k^∞: $R_{K/k}(\mathbf{G}_m)_k^\infty \longrightarrow (G \times \mathbf{G}_m)_k^\infty$, and $(\hat{\lambda})_k$: $\widehat{(G \times \mathbf{G}_m)}_k \longrightarrow \widehat{R_{K/k}(\mathbf{G}_m)}_k$ (cf. [2]), ∞ being the set of all infinite primes of k. For a

AMS (MOS) subject classifications (1970). Primary 12A50, 20G30.

[1] For the unexplained notions, see [2], [3], [5].