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ON THE ISOMETRIES OF $L^p(\Omega, X)$

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The isometries of $L^p[0, 1]$, $1 \leq p < \infty$, $p \neq 2$, were determined by Banach [1, p. 178]. In that case every isometry T is of the form $(Tf)(\cdot) = u(\cdot)f(\phi(\cdot))$ where ϕ is a measurable transformation of $[0, 1]$ onto itself and u is a fixed function related by ϕ by the formula $|u|^p = d(\lambda \circ \phi)/d\lambda$ where λ is Lebesgue measure. Lamperti [4] determined the isometries of $L^p(\Omega)$ for any σ-finite measure space (Ω, Σ, μ). The result resembles Banach's except for the replacement of the point transformation ϕ by a set transformation. Cambern [3] determined the surjective isometries of $L^p(\Omega, K)$ for a separable Hilbert space K. These isometries resemble those of $L^p(\Omega)$ except for the emergence of an operator-valued function.

Our aim here is to describe the surjective isometries of $L^p(\Omega, X)$ for certain Banach spaces X (Theorem 1) and the *injective* isometries of $L^p(\Omega, K)$ for a separable Hilbert space K (Theorem 2).

Let (Ω, Σ, μ) be a σ-finite measure space. A *set homomorphism* Φ is a map of Σ into itself, defined modulo null sets, which preserves set differences and countable unions. If, in addition, $\mu(\Phi(\delta)) = 0$ if and only if $\mu(\delta) = 0$, then Φ is called a *set isomorphism*. It can be shown that Φ induces a transformation, also denoted by Φ, on the space of measurable functions defined on Ω with values in a separable Banach space X.

A Banach space X is called the *l^p-direct sum* of two Banach spaces X_1 and X_2 if X is isometrically isomorphic to $X_1 \oplus X_2$ with $\|x_1 \oplus x_2\|^p = \|x_1\|^p + \|x_2\|^p$.

THEOREM 1. *Let T be an operator on $L^p(\Omega, X)$, $1 \leq p < \infty$, $p \neq 2$, where X is a separable Banach space, and assume that X is not the l^p-direct sum of two nonzero Banach spaces (for the same p). Then T is a surjective isometry if and only if*

$$(1) \qquad (Tf)(\cdot) = S(\cdot)h(\cdot)(\Phi(f))(\cdot) \qquad f \in L^p(\Omega, X),$$

where Φ is a set isomorphism of the measure space onto itself, S is a strongly measurable map of Ω into $B(X)$ with $S(t)$ a surjective isometry of X for almost all $t \in \Omega$, and $h = (d\nu/d\mu)^{1/p}$ where $\nu(\cdot) = \mu(\Phi^{-1}(\cdot))$.

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