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NONLINEAR ERGODIC THEOREMS

BY H. BRÉZIS AND F. E. BROWDER

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In two recent notes ([1], [2]), J-B. Baillon proved the first ergodic theorems for nonlinear mappings in Hilbert space. We simplify the argument here and obtain an extension of Baillon's theorems from the usual Cesàro means of ergodic theory to general averaging processes $A_n = \Sigma_{k=0}^{\infty} a_{n,k} T^k$ $(0 \leqslant a_{n,k}, \Sigma_{k \geqslant 0} a_{n,k} = 1)$.

THEOREM 1. *Let H be a Hilbert space, C a closed bounded convex subset of H, T a nonexpansive self map of C. Suppose that as $n \longrightarrow \infty$, $a_{n,k} \longrightarrow 0$ for each k, and $\gamma_n = \Sigma_{k=0}^{\infty}(a_{n,k+1} - a_{n,k})^+ \longrightarrow 0$. Then for each x in C, $A_n x = \Sigma_{k=0}^{\infty} a_{n,k} T^k x$ converges weakly to a fixed point of T.*

The proof of Theorem 1 depends upon an extension of Opial's lemma [3].

LEMMA 1. *Let $\{x_k\}$ and $\{y_k\}$ be two sequences in H, F a nonempty subset of H, C_m the convex closure of $\bigcup_{j \geqslant m} \{x_j\}$. Suppose that*

(a) *For each f in F, $|x_j - f|^2 \longrightarrow p(f) < +\infty$;*

(b) $\operatorname{dist}(y_k, C_m) \longrightarrow 0$ *as $k \longrightarrow \infty$ for each m;*

(c) *Any weak limit of an infinite subsequence of $\{y_k\}$ lies in F.*

Then y_k converges weakly to a point of F.

PROOF OF LEMMA 1. Since $\{y_k\}$ is bounded, it suffices to show that if f and g in F are weak limits of infinite subsequences of $\{y_k\}$, then $f = g$. For each j,

$$|x_j - f|^2 = |x_j - g|^2 + |g - f|^2 + 2(x_j - g, g - f).$$

For a given $\epsilon > 0$, there exists $m(\epsilon)$ such that for $j \geqslant m(\epsilon)$,

$$|p(g) - |x_j - g|^2| < \epsilon; \qquad |p(f) - |x_j - f|^2| < \epsilon.$$

Let K_ϵ be the convex set of all u such that

$$|2(u - g, g - f) + p(g) - p(f) + |g - f|^2| \leqslant 2\epsilon.$$

Since K_ϵ contains $\bigcup_{j \geqslant m(\epsilon)} \{x_j\}$, it contains $C_{m(\epsilon)}$. There exists k_ϵ such that for $k \geqslant k_\epsilon$ we can find u_k in $C_{m(\epsilon)}$ such that $|y_k - u_k| \leqslant \epsilon$. For $k \geqslant k_\epsilon$, it follows that

$$|2(y_k - g, g - f) + p(g) - p(f) + |g - f|^2| \leqslant 2\epsilon + 2\epsilon|g - f|.$$

Consider an infinite subsequence $\{y_{k_s}\}$ for which $(y_{k_s} - g, g - f) \longrightarrow 0$. In the limit

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