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GLOBAL RESULTS IN CONTROL THEORY WITH APPLICATIONS TO UNIVALENT FUNCTIONS

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1. A problem in control theory. Many classical coefficient problems in the theory of univalent functions can be stated as the following control problem. Consider a first order differential system

$$dx/dt = f(x, u(t)),$$

where $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$ and $f(x, u) = (f_1(x, u), \ldots, f_n(x, u))$ are real valued vectors. Assume that f is continuous on $R^n \times R^m$ and for fixed u, $f \in C^1(R^n)$. The values of $u(t)$ are in a compact domain $\mathcal{U} \subset R^m$. Denote by $\mathcal{F}$ the class of all piecewise continuous functions $u(t)$ for $t \geqslant 0$ with the values in $\mathcal{U}$. Let $x(t)$ satisfy a fixed initial condition $x(0) = \xi$. Denote by $x(t, u)$ the solution of the system above for a given $u(t)$ in $\mathcal{F}$. Let $F(x) = F(x_1, \ldots, x_n)$ belong to $C^1(R^n)$.

THEOREM 1. *Let $u^* = u^*(t)$ be a solution of the problem* $\sup_{\mathcal{F}} F(x(T, u)) = F(x(T, u^*))$, *for $T > 0$. Consider the system*

$$dx/dt = f(x, u^*(t)), \quad x(\tau) = \eta$$

for $0 \leqslant \tau \leqslant T$. Define a function F_τ by the equality $F_\tau(\eta) = F(x(T))$. Then $x(\tau, u^*)$ solves the problem $\sup_{\mathcal{F}} F_\tau(x(\tau, u)) = F_\tau(x(\tau, u^*))$.

The proof of the theorem follows by considering the functions $u(t)$ such that $u(t) = u^*(t)$ for $\tau < t \leqslant T$. In case where $f(x, u) = A(u)x$ and $F(x) = \lambda_0' x$ Theorem 1 has a very simple form. Here $A(u) = (a_{ij}(u))_1^n$ and $a_{ij}(u) \in C(R^m)$. By A' and λ' we denote the corresponding transposed matrix and vector.

THEOREM 2. *Consider a control system $dx/dt = A(u(t))x$. Let $u^*(t)$ solve the linear problem*

$$\sup_{\mathcal{F}} \lambda_0' x(T, u) = \lambda_0' x(T, u^*).$$

Then $x(\tau, u^*)$ solves the linear problem

$$\sup_{\mathcal{F}} \lambda'(\tau) x(\tau, u) = \lambda'(\tau) x(\tau, u^*),$$

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