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ON A NEW DEFINITION OF DERIVATIVE

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It is well known, since Weierstrass, that there are continuous functions that are not derivable at any point. The same is true for the various known generalizations of derivative, e.g. the unilateral, approximate and the symmetric derivatives. The present announcement deals with a new definition of derivative in terms of which every continuous function f is derivable at a *c-dense* set of points (viz. the set meets every interval in a set whose power is c), and the properties of f can be investigated in terms of the values of its new derivative wherever it exists.

Let f: $R \longrightarrow R$, where R denotes the set of real numbers. Let f be called *upper derivable* at a point $x \in R$ if $D^+f(x) \leqslant D_-f(x)$, and then an extended real number α is called an *upper derivative* of f at x if $D^+f(x) \leqslant \alpha \leqslant D_-f(x)$. Defining f to be *lower derivable* at x if $-f$ is upper derivable at x, it is clear that f is derivable at x if and only if it is upper and lower derivable there. These definitions can be easily extended to real-valued functions on any real topological vector space.

What is unusual about the upper and lower derivatives is that they are not unique like the derivative; consider e.g. $f(x) = |x|$ at $x = 0$. They are, however, unique at all but a countable set of points. Also, if f is *nonangular,* viz. $D_-f \leqslant D^+f$ and $D_+f \leqslant D^-f$ everywhere, then the upper or lower derivative of f is unique at every point where it exists. Such functions, in fact, form a residual set in the space $C[0, 1]$ with the uniform norm.

If f has a finite upper derivative at a point x, it is clearly upper semicontinuous at x. Every u.s.c. function f, on the other hand, has a finite upper derivative at a dense set of points, and this set becomes c-dense when f is nonangular. With the help of an analogue [1] of the Denjoy-Young-Saks theorem, we prove

THEOREM 1. *If f is continuous, then for almost every value of y in R, the level $f^{-1}(y)$ contains two dense sets of points where f has unique upper and*

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