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ON THE GALOIS GROUPS OF CUBICS AND TRINOMIALS

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A polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \qquad (a_i \in \mathbf{Z}) \tag{1}$$

is said to be "with affect" if the Galois group G_f of its splitting field, considered as a permutation group on the roots of $f(x)$, is a proper subgroup of the symmetric group on n letters. The purpose of this note is to give an improved upper bound for the number of monic polynomials of degree three with affect. More generally, we also give an upper bound for the number of trinomials f of degree n for which G_f is a subgroup of the alternating group A_n on n letters. Details will appear elsewhere [6].

Previous upper bounds for the number $E_n(N)$ of monic polynomials of degree n with integer coefficients bounded in absolute value by N, which are with affect, have been given by van der Waerden [10], Knobloch [5], and Gallagher [2]. The best estimate obtained so far for the general case is $E_n(N) \ll N^{n-1/2}\log N$ given by Gallagher [2]. Van der Waerden [10] has conjectured that $E_n(N) \ll N^{n-1}$, a bound which does hold for the reducible polynomials if $n \geqslant 3$ [10]. Our results are as follows.

THEOREM 1. *For each* $\epsilon > 0$, $E_3(N) \ll_\epsilon N^{2+\epsilon}$.

THEOREM 2. *Let* $J_{k,n}(N)$ *denote the number of trinomials*

$$f(x) = ax^n + bx^k + c \tag{2}$$

with $a, b, c \in Z$ *and* $|a|, |b|, |c| \leqslant N$, *for* $N \geqslant 1$, *for which* G_f *is a subgroup of* A_n. *Then for each* $\epsilon > 0$, $J_{k,n}(N) \ll_{\epsilon,n} N^{2+\epsilon}$.

The proofs of these two theorems are similar. It is well known that the discriminant D_f of (1) has the property that $D_f \in \mathbf{Z}[a_0, a_1, \ldots, a_n]$. Moreover, if $G_f \subseteq A_n$, then D_f is the square of a rational integer, i.e.

$$D_f = z^2 \qquad (z \in \mathbf{Z}). \tag{3}$$

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