BANACH LATTICES WITH A WEAK ORDER UNIT

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Starting from a result due to H. Rosenthal (see Lemma 1.3 in [6]), we present a new result which outlines a geometrical condition for the existence of a weak order unit (i.e., a total element) in the dual of a Banach lattice: the non-existence of a lattice isomorph of a space $l_1(\Gamma)$ for Γ an uncountable set (see Theorem 2 below). As a consequence we obtain that the dual of a Banach space E having local unconditional structure (as defined in [1] has the Radon-Nikodým property iff E does not contain a copy of l_1. R. C. James constructed, in [3], an example of separable Banach space J with a nonseparable dual such that c_0 and l_1 do not embed in J. Consequently J does not have local unconditional structure.

Let L be the class of all Banach spaces X satisfying the following two conditions:

($L1$) X' is an order complete Banach lattice;

($L2$) there is a $v \in X''$ such that $x \in X$, $|x| \wedge |v| = 0$ implies $x = 0$.

Each Banach lattice with a weak order unit, or each predual of an $L_1(\mu)$ space, belongs to L.

1. LEMMA. *If $X \in L$, then there exists an order complete Banach lattice E with a weak order unit and a lattice isometry $i\colon X' \to E'$ such that: (a) $i(X')$ is complemented in E', and (b) $i(X')$ is formed by order continuous functionals on E.*

Hint. Consider for E the band generated by v in X''.

If Z is an order complete Banach lattice and $A \subset Z$ is a closed subspace, then we shall denoted by $\Sigma(A)$ an order complete closed sublattice of the band generated by A such that $A \subset \Sigma(A)$.

2. THEOREM. *Let $E \in L$ and let $A \subset E'$ be a closed subspace. Then either*:

(i) $\Sigma(A)$ *has a weak order unit* $u' > 0$; *or,*

(ii) *A contains an isomorph of $l_1(\Gamma)$ (for Γ an uncountable set) that is complemented in $\Sigma(A)$, and $\Sigma(A)$ contains a lattice isomorph of $l_1(\Gamma)$.*