BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

ON IDEALS OF SETS AND THE POWER SET OPERATION

BY THOMAS JECH[1] AND KAREL PRIKRY[2]

Communicated by S. Feferman, February 16, 1976

We present some inequalities involving cardinal powers. In most of the results we assume the existence of an ideal I satisfying a weak completeness condition.

For the remainder of this paper, *I will always denote an ideal over ω_1 containing all enumerable sets.* $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$ is *I-disjoint* if $X \cap Y \in I$ for all distinct $X, Y \in \mathcal{F}$; $\mathcal{F}$ is *almost disjoint* if $|X \cap Y| \leqslant \aleph_0$ for all distinct $X, Y \in \mathcal{F}$. I is *λ-saturated* if $|\mathcal{F}| < \lambda$ for every I-disjoint $\mathcal{F} \subseteq \mathcal{P}(\omega_1) - I$.

THEOREM 1. *Let I be σ-additive. If $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$, then for every $\lambda < 2^{\aleph_1}$ there is an almost disjoint $\mathcal{F} \subseteq \mathcal{P}(\omega_1) - I$ with $|\mathcal{F}| = \lambda$. Moreover, if $2^{\aleph_1}$ is singular, we get such an $\mathcal{F}$ with $|\mathcal{F}| = 2^{\aleph_1}$. Hence if $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} < \aleph_{\omega_1}$, then there exists no λ-saturated ideal for any $\lambda < 2^{\aleph_1}$.*

REMARK. In [1] the same assumption on $2^{\aleph_0}$ is used to obtain an almost disjoint $\mathcal{F}$ such that $|\mathcal{F}| = 2^{\aleph_1}$. In [3] stronger assumptions on $2^{\aleph_0}$ are used to show that the ideal of nonstationary sets is not $\aleph_2$-saturated.

For $S \in \mathcal{P}(\omega_1) - I$, $\mathcal{W}$ is an *I-partition* of S if $\mathcal{W}$ is a maximal I-disjoint family $\subseteq \mathcal{P}(S) - I$. If $\mathcal{W}_0$ and $\mathcal{W}_1$ are I-partitions of S, then $\mathcal{W}_1$ is a refinement of $\mathcal{W}_0$ if every $X \in \mathcal{W}_1$ is included in some $Y \in \mathcal{W}_0$.

I is *precipitous* if for every $S \in \mathcal{P}(\omega_1) - I$, and every sequence $\mathcal{W}_n$ $(n \in \omega)$ of I-partitions of S such that $\mathcal{W}_{n+1}$ is a refinement of $\mathcal{W}_n$, there exists a sequence $X_n \in \mathcal{W}_n$ such that $X_{n+1} \subseteq X_n$ and $\bigcap \{X_n: n \in \omega\} \neq 0$.

PROPOSITION. *If there is a precipitous I, then there is a σ-additive, normal, precipitous I. If I is normal and precipitous, then ω_1 is measurable in $L[I]$. If I is $\aleph_2$-saturated, then I is precipitous. The ideal $\{X \subseteq \omega_1: |X| \leqslant \aleph_0\}$ is not precipitous.*

We shall consider a class of cardinal functions called *nice functions.* The following functions are nice: $\Phi(\alpha) = \omega_\alpha$; $\Phi(\alpha) =$ the αth weakly inaccessible cardinal. If Φ and ψ are nice, then so are, for example, $\psi_1(\alpha) =$ the αth fixed point of Φ; $\psi_2(\alpha) = \Phi(\alpha + \alpha)$; $\psi_3(\alpha) = \Phi(\psi(\alpha))$.

AMS (MOS) subject classifications (1970). Primary 02K35.
[1]Research supported by NSF Grant MPS75-07408.
[2]Research supported by NSF Grant GP-43841.
Copyright © 1976, American Mathematical Society