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# A SPECTRAL THEOREM FOR NONLINEAR OPERATORS

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The most useful form of the spectral theorem asserts that every bounded normal operator on a complex Hilbert space is unitarily equivalent to a multiplication operator. In this note we report a (verbatim) generalization of that theorem to nonlinear operators. While the need for this result arose in attempting to define "spectral" invariants for nonlinear random processes [1], it appears to be a basic statement in operator theory which may be applicable elsewhere.

Let $\mathcal{H}$ and $\mathcal{K}$ be (separable) complex Hilbert spaces. By a *bounded holomorphic* operator from $\mathcal{H}$ to $\mathcal{K}$ we mean a mapping $F$: $\mathcal{H} \longrightarrow \mathcal{K}$ which satisfies

(i) $\sup_{\|z\| \leq 1} \|F(z)\| < \infty$,

(ii) for every $z_1, \ldots, z_n$ in $\mathcal{H}$ and every $w \in \mathcal{K}$,

$$(F(a_1 z_1 + \cdots + a_n z_n), w)$$

defines an entire function of the $n$ complex variables $a_1, \ldots, a_n$.

Every linear operator is of course holomorphic. More generally, if $F$ is a *monomial* in the sense that it has the form $F(z) = G(z, z, \ldots, z)$, where $G$ is a bounded $n$-linear operator from $\mathcal{H} \times \cdots \times \mathcal{H}$ into $\mathcal{K}$, then $F$ is holomorphic.

Linear spectral theory applies to operators from $\mathcal{H}$ into itself. In the nonlinear (holomorphic) case, the appropriate range space is not $\mathcal{H}$ but the Fock space over $\mathcal{H}$, which we will write as $e^{\mathcal{H}}$:

$$e^{\mathcal{H}} = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \cdots,$$

where $\mathcal{H}^0 = \mathbf{C}$ and $\mathcal{H}^n$ is the symmetric Hilbert space tensor product of $n$ copies of $\mathcal{H}$, $n \geq 1$. A key feature of this construction is that there is a natural representation $\pi$ of the full unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ as unitary operators on $e^{\mathcal{H}}$. For each unitary $U$ on $\mathcal{H}$, $\pi(U)$ is defined as $U^0 \oplus U^1 \oplus U^2 \oplus \cdots$, where $U^0 = 1$ and $U^n$ is the $n$-fold tensor product of copies of $U$. This representation has been studied in some detail by Irving Segal in connection with the mathematical description of quantum systems having infinitely many degrees of freedom [2], [3]. It also plays an essential role in the following considerations.

A linear operator is normal iff it belongs to an abelian von Neumann algebra. This property can be generalized to bounded holomorphic operators $F$: $\mathcal{H} \longrightarrow e^{\mathcal{H}}$ as follows. We will say that $F$ *commutes* with a unitary $U$ in $\mathcal{U}(\mathcal{H})$ if

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