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ON STOPPING TIME DIRECTED CONVERGENCE

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The main purpose of this note is to introduce the notion of $\bar{S}$-martingales, a certain modification of that of asymptotic martingales, the main justification of which is III.

1. **S-convergence.** Let (Ω, F, P) be a probability; (F_n) $(n = 1, 2, \ldots)$, a nondecreasing sequence of measurable σ-fields and (X_n) an adapted sequence of extended real-valued r.v. (random variables). (If the F_n are not mentioned explicitly then any F_n with the above properties will do; in particular, we may take F_n to be the σ-field generated by $X_1, \ldots, X_n$.) Let $T = \{t\}$ be the family of *bounded stopping times*; i.e. the family of positive, bounded, integer-valued r.v. t with $t^{-1}(n) \in F_n$ for all n. T is a directed set filtering to the right under the relation $t_1 \leqslant t_2$, i.e. $t_1(\omega) \leqslant t_2(\omega)$ a.s. (almost surely). The r.v. X_t for $t \in T$, is defined by $X_t(\omega) = X_{t(\omega)}(\omega)$.

DEFINITION. Let ϕ map X_t $(t \in T)$ into a topological space M. Then $(\phi(S_n))$ is said to be *S-convergent*—or stopping time directed convergent—(to Y) if the directed set $\phi(X_t)$ is convergent in the topology of M (to Y).

S-convergence implies ordinary convergence, but not vice-versa.

EXAMPLES. (1) ϕ the identity mapping, M the space of all extended real valued r.v. topologized by convergence in probability (for extended real valued r.v. this is interpreted as applied to the r.v. obtained through the mapping $x \longrightarrow x/(1 + |x|)$). We then speak of *S-convergence in probability*. In sharp distinction from the situation in ordinary convergence, we have

I. *S-convergence in probability is equivalent to a.s. convergence.*

The proof is immediate since there exist $t_1 < \cdots < t_n < \cdots \longrightarrow \infty$ with $X_{t_n} \xrightarrow{\text{a.s}} \limsup X_{t_n} = \limsup X_n$.

(2) (X_n) is said to be an *S-martingale* if the expectations (finite or not) EX_t are defined for all $t \in T$ and (EX_n) is S-convergent (to a finite or infinite number). If the limit is a finite number then (X_n) is called an *asymptotic martingale.*

The argument proving I yields

II. *A uniformly bounded sequence of r.v. (X_n) is a.s. convergent iff it is an asymptotic martingale.*

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