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SINGULAR PERTURBATION THEORY FOR SEMIBOUNDED OPERATORS

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In this announcement an operator theoretic approach to singular perturbation expansions for simple eigenvalues is outlined. Corresponding results hold for eigenvectors, eigenvalues of finite multiplicity, and spectral concentration near eigenvalues of finite multiplicity.

We seek first and second order approximations for problems to which the regular perturbation method does not apply (cf. [2]). First presented are the abstract singular perturbation expansions, followed by mention of some results for differential problems. Application of the abstract results to concrete problems is involved, but our framework makes Lions' method of correctors [3], as well as boundary layer techniques (cf. [4]), applicable to eigenvalue problems.

Let H be a complex Hilbert space with inner product (v, w) and norm $|v|$. Let $b(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(b)$ which is dense in H. We assume that the quadratic form corresponding to $b(v, w)$ has a positive lower bound, and is closed. Then $D(b)$, with inner product $b(v, w)$, is a Hilbert space. Further let $a(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(a)$ which is dense in $D(b)$, and assume that the quadratic form corresponding to $a(v, w)$ is nonnegative, and closed in $D(b)$.

Let B be the positive definite selfadjoint operator in H defined by $(Bv, w) = b(v, w)$ for all $w \in D(b)$, on $D(B) = \{v \in D(b)\colon w \longrightarrow b(v, w)$ is continuous on $D(b)$ in the topology induced by $H\}$. Similarly let A_ϵ, $\epsilon > 0$, be the positive definite selfadjoint operator in H defined by $(A_\epsilon v, w) = \epsilon a(v, w) + b(v, w)$, and A the nonnegative selfadjoint operator in $D(b)$ defined by $b(\mathsf{A}v, w) = a(v, w)$.

Assume that λ is an isolated simple eigenvalue of B with corresponding eigenvector u normalized in H. Assume further that λ is stable under the above perturbation, i.e., that for ϵ sufficiently small the intersection of any isolating interval for λ and the spectrum of A_ϵ consists of a single simple eigenvalue λ_ϵ of A_ϵ, $\lambda_\epsilon \longrightarrow \lambda$ as $\epsilon \downarrow 0$ (cf. [2]).

Now let $\eta = (A_\epsilon^{-1}u, u)$. It is easily shown that $\eta = \lambda^{-1} - \epsilon\lambda^{-2}\lambda'_\epsilon$, where $\lambda'_\epsilon = b(\mathsf{A}(\epsilon\mathsf{A} + I)^{-1}u, u)$ and that $\epsilon\lambda'_\epsilon \longrightarrow 0$ as $\epsilon \downarrow 0$ (cf. [1]).

THEOREM 1. (i) $\lambda_\epsilon = \lambda + O(\epsilon\lambda'_\epsilon)$ *as* $\epsilon \longrightarrow 0$.

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