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THE FUNCTIONAL EQUATION
$af(ax) + bf(bx + a) = bf(bx) + af(ax + b)$
EXTENSIONS AND ALMOST PERIODIC SOLUTIONS

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Composition operators play a central role in the study of many naturally defined classes of linear transformations. They are often extremal elements in a geometric or analytic sense. A simple class of such operators acting on $L^1(I)$, where $I = [0, 1]$ with Lebesgue measure, consists of those determined by composition with measure preserving transformations $\phi: I \longrightarrow I$. The operator T_ϕ is then defined by $T_\phi f = f \circ \phi$. If we consider the convex class $\mathcal{D}$ of all operators defined on $L^1(I)$ which satisfy (i) $T1 = 1$, (ii) $\int_I Tf = \int_I f$, (iii) $Tf \geqslant 0$ whenever $f \geqslant 0$, then the composition operators are among the extreme points of $\mathcal{D}$ [1]. Closely associated with such a convex class of operators are the orbits Ω of various elements of $L^1(I)$: $\Omega(f) = \{Tf\colon T \in \mathcal{D}, f \text{ fixed in } L^1(I)\}$. The extreme points of each orbit are known [2], [3] and the question arises (applicable to other classes of operators as well):

Does there exist an extreme operator in $\mathcal{D}$ which preserves no extreme points of any orbit? That is, does there exist an extreme element of $\mathcal{D}$ such that whenever g is an extreme point of $\Omega(f)$, then Tg is not extreme? This should hold true for all nonconstant $f \in L^1(I)$ (see [5]).

It is enough to show that an extreme operator can be found which preserves no characteristic functions except for the constant functions 1 and 0 [4, Lemma 4]. To this end we consider two (noninvertible) measure preserving transformations:

$$\phi(x) = \begin{cases} x/a, & 0 \leqslant x \leqslant a, \\ (x-a)/b, & a \leqslant x \leqslant 1, \end{cases} \qquad \psi(x) = \begin{cases} x/b, & 0 \leqslant x \leqslant b, \\ (x-b)/a, & b \leqslant x \leqslant 1, \end{cases}$$

where $a + b = 1$. In order to define a mapping on $L^1(I)$ which fails to preserve characteristic functions, choose a function γ, $0 < \gamma < 1$, and set

$$T = \gamma T_\phi + (1 - \gamma)T_\psi. \tag{1}$$

We will leave the class $\mathcal{D}$ unless γ is also chosen so that (ii) is satisfied. This

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