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LINEAR ANALOGUES OF ULTRAFILTERS

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1. **Introduction.** The automorphism group of a vector space V acts on any object constructed naturally from V, and the corresponding orbit space is often of considerable mathematical interest. Indeed many classical algebraic questions can be viewed in this way. In this paper we consider the (classically uninteresting) case when the object is the second dual V^{**} of V. The motivation for this is threefold: first, it arises naturally in the above context; second, it is essentially a linear analogue of the study of types of ultrafilters; and third, it has applications to structure theory for infinite abelian groups and primitive rings with minimal left ideals [2].

We present here a statement of some of our results. Details and further results will appear elsewhere.

2. **The orbit space.** Let F be a field and let V be a vector space over F. Then V is naturally embedded in its second dual V^{**}. Furthermore any endomorphism φ: $V \longrightarrow V$ extends naturally to an endomorphism φ^{**}: $V^{**} \longrightarrow V^{**}$ Hence the action of the general linear group $\mathrm{GL}(V)$ on V extends to an action of $\mathrm{GL}(V)$ on V^{**}, and we can consider the orbit space $V^{**}/\mathrm{GL}(V)$. If V is finite dimensional, then $V = V^{**}$ and the orbit space contains just two elements $\bar{o}$ and $\bar{v}$, where "bar" denotes projection and v is any nonzero element of V. If V is infinite dimensional, however, then V^{**} is much larger than V and the orbit space is highly nontrivial.

Henceforth we consider only the case when F is finite and V has countable dimension. (In the applications F is usually a prime field.) Then V has cardinality $\aleph_0$, V^* has cardinality $2^{\aleph_0}$, and V^{**} has cardinality $2^{(2^{\aleph_0})}$. Since $\mathrm{GL}(V)$ has cardinality $2^{\aleph_0}$ it is immediate that the orbit space has cardinality $2^{(2^{\aleph_0})}$. We wish to analyze the structure of this orbit space, and in particular to determine its dependence on the field F.

3. **The partial order.** There is a natural partial order on the orbit space $V^{**}/\mathrm{GL}(V)$ which is analogous to the Rudin-Keisler order on ultrafilters. If λ_1, λ_2 are in V^{**} we define $\lambda_1 \geqslant \lambda_2$ if and only if $\varphi^{**}(\lambda_1) = \lambda_2$ for some endomorphism φ of V. The relation $\geqslant$ is clearly reflexive and transitive. Furthermore, if λ_1 and λ_2 lie in the same orbit, then $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_1$. The converse fails, however,

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