BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, March 1976

EXAMPLES OF ELLIPTIC COMPLEXES

BY R. T. SMITH

Communicated by François Treves, October 16, 1975

The main purpose of this note is to give natural geometric examples of elliptic complexes for which the Poincaré lemma fails. Indeed:

(a) There are natural (and even involutive) elliptic complexes which are not formally exact, and whose local cohomology is infinite (Examples 2, 3). On the other hand:

(b) An arbitrary locally exact elliptic complex need not be formally exact (cf. Example 4′).

These remarks reflect interestingly on the outstanding problem in the theory (Spencer's conjecture): Is a formally integrable formally exact elliptic complex locally exact? (See Goldschmidt [2] for a complete analysis of the formal theory.) Thus (a) demonstrates forcibly the *independence* of the hypotheses, whereas (b) shows that the hypothesis of formal exactness is not always *necessary*.

Most of our examples take the following form: Let E be a subbundle of $\Lambda^p(\mathbf{R}^{n*})$; let $\underline{E}$ denote the sheaf of germs of sections of E. Then there are complexes of the following types:

$$\text{(I)} \qquad \underline{\Lambda}^{p-2} \xrightarrow{d} \underline{\Lambda}^{p-1} \xrightarrow{\pi d} \underline{\Lambda^p/E};$$

$$\text{(II)} \qquad \underline{E} \xrightarrow{d|E} \underline{\Lambda}^{p+1} \xrightarrow{d} \underline{\Lambda}^{p+2}.$$

Note to begin with that the cohomology of (I) is equivalent to the space of closed sections of E, i.e., the solution space of a *homogeneous* system of equations. One of our basic observations is then:

(c) There are nontrivial examples of these types which are elliptic (cf. Examples 2, 3).

On the other hand, Spencer's conjecture itself cannot be disproved within the context of such examples: if E is nontrivial, (I) is not formally exact; if (II) is elliptic (no further hypotheses), one checks it is locally exact.

Constant coefficient examples.

EXAMPLE 1 (NIRENBERG). An arbitrary elliptic complex need not be formally or locally exact. Over $\mathbf{C}^n$ construct

---

AMS (MOS) subject classifications (1970). Primary 35N05, 35N10; Secondary 58G05.
Key words and phrases. Elliptic complex, formally exact complex, Dirac complex, involutive operator.