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ITERATED LOOP FUNCTORS AND THE HOMOLOGY OF THE STEENROD ALGEBRA

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Let A be the mod-2 Steenrod algebra. For any unstable A-module M the "unstable homology groups" $H^A_{s,k}(M) = \mathrm{Tor}^A_{s,k}(M)$ are defined by means of unstable projective resolutions of M [2]. We describe here a new approach to the problem of computing these groups.

Let $\mathcal{M}_A$ be the category whose objects are unstable A-modules and whose morphisms are degree preserving A-maps. For M in $\mathcal{M}_A$ and x in M_n we write, as is usual $\mathrm{Sq}_a x = \mathrm{Sq}^{n-a}x$. Let "suspension" $S\colon \mathcal{M}_A \longrightarrow \mathcal{M}_A$ be the functor that raises degree by 1. S has a left adjoint $\Omega\colon \mathcal{M}_A \longrightarrow \mathcal{M}_A$ [2] given by $(\Omega M)_n = (\mathrm{coker}\, \mathrm{Sq}_0)_{n+1}$, with A-action induced by that on M. The left derived functors Ω_s ($s \geq 0$) of Ω are defined in the usual way: given M in $\mathcal{M}_A$ one forms a projective resolution $\cdots \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow M \longrightarrow 0$. Then $\Omega_s M$ is the sth homology group of the complex $\cdots \longrightarrow \Omega P_1(M) \longrightarrow \Omega P_0(M) \longrightarrow 0$. The left derived functors of Ω are completely understood [1], [2], [3]. In fact,

$$\Omega_s M = 0 \quad \text{if } s > 1, \tag{1}$$

$$(\Omega_1 M)_{2n-1} = (\ker \mathrm{Sq}_0)_n \tag{2}$$

with A-action given by $\mathrm{Sq}_a \Omega_1 x = \Omega_1 \mathrm{Sq}_{(a+1)/2}$ for x in $\ker \mathrm{Sq}_0$.

Consider now the k-fold iterate Ω^k of Ω. We pose:

PROBLEM (*). Give a workable description of the left derived functors Ω^k_s of Ω^k, for all $s \geq 0$.

Our interest in these derived functors stems from the fact that their zero-dimensional components are the unstable homology groups of the Steenrod algebra:

THEOREM 1. *There is a natural isomorphism* $\mathrm{Tor}^A_{s,k}(M) = (\Omega^k_s M)_0$.

Our interest in Problem (*) is heightened by the fact that it appears to be solvable: there is a simple relation between the derived functors of Ω^k and those of Ω^{k-1}.

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