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ON THE NONTRIVIALITY OF SOME GROUP EXTENSIONS GIVEN BY GENERATORS AND RELATIONS

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Let G be any group, $F = F(x_1, x_2, \ldots, x_n)$ the free group on n generators. Consider the group presentation $H = G*F/R_1, R_2, \ldots, R_n$, where each relation R_i is a product of conjugates, by elements of G, of elements of F:

$$R_i = \prod_v g(i, v) x_{\alpha(i,v)}^{r(i,v)} g^{-1}(i, v).$$

Then G injects into H, and we want to know when H is genuinely larger than G. A criterion will be framed in terms of the Fox matrix, E, of the presentation:

$$(E)_{i,j} = \sum_v r(i, v) g(i, v)$$

where the sum is over those v for which $\alpha(i, v) = j$. The assumption that $H = G$ implies by a formal argument (or use of Fox free differential calculus) that E is invertible matrix over $\mathbf{Z}(G)$; to avoid this trivial case, we will assume $E \in \mathrm{GL}(n, \mathbf{Z}(G))$.

E may equally well be written $E = \Sigma M(g)g$, each $M(g) \in M(n, \mathbf{Z})$. For any finite dimensional representation ρ of G we define $\rho(E) = \Sigma M(g) \otimes \rho(g)$. Note that $\det 1(E) = \pm 1$.

Let A be the subgroup of $\mathrm{GL}(n, \mathbf{R}(G))$ generated by squares and commutators. Our main result is

THEOREM 1. *Assume G finite,* $\det 1(E) = 1$, *and n odd. If $E \notin \bigcup_{g \in G} \{Ag\}$ then G injects properly into H.*

(The case n even can be reduced to the preceding by adding a free generator to H and a relation which kills it.)

The proof needs several preliminary considerations. Let L be a compact connected Lie group of rank m, dimension d, and $\mathfrak{L}$ its Lie algebra. Let φ be any homomorphism of G into L; $\mathrm{Ad}\,\varphi$ is then a representation of G on $\mathfrak{L}$. By $\varphi(R_i)$ we mean the relation R_i with elements of G therein occurring replaced by their images under φ. Consider the map $f: L^n \longrightarrow L^n$ given by $f_i(x_1, x_2, \ldots, x_n) = \varphi(R_i) \cdot x_i$. The identity element of L^n is a fix point of f; another fix point of f assures H is larger than G.

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