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PARTIAL AND COMPLETE CYCLIC ORDERS

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We show that, in contrast to a famous theorem on linear orders, not every partial cyclic order on $M = \{1, \ldots, m\}$ can be extended to a complete cyclic order. In fact, the complexity, in a certain sense, of sufficient conditions for such an extendability increases rapidly with m.

DEFINITION 1. (i) Two linear orders, $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$, on M are called *cyclically equivalent* if there exists $k \in M$ such that $[j - 1 \equiv (i - 1 + k) \pmod m] \Rightarrow a_i = b_j$.

(ii) A *complete cyclic order* (CCO) on M is an equivalence class C of linear orders modulo cyclic equivalence; denote $a_1 a_2 \cdots a_m$ for the equivalence class containing $(a_1, a_2, \ldots, a_m)$.

DEFINITION 2. A *partial cyclic order* (PCO) on M is a set Δ of cyclically ordered triples (COTs) out of M such that:

(i) $xyz \in \Delta \Rightarrow zyx \notin \Delta$ ("antisymmetry"),

(ii) $\{xyz, xzw\} \subset \Delta \Rightarrow xyw \in \Delta$ ("transitivity");

since $xyz = zxy$, etc., also $yzw \in \Delta$ is implied.

THEOREM 3. (i) *If C is a CCO then the set Δ of all COTs derived from C is a PCO.* (ii) *If Δ is a saturated PCO, i.e., $\{x, y, z\} \in \binom{M}{3}$ & $xyz \notin \Delta \Rightarrow zyx \in \Delta$, then there exists a CCO from which all of Δ's COTs are derived*; *Δ is then said to be extendable to a CCO.*

COROLLARY 4. *A PCO is extendable to a CCO if and only if it is contained in a saturated PCO.*

It is natural to ask whether every PCO is extendable to a CCO (or, equivalently, is contained in a saturated PCO). In view of the following example, the answer is in the negative.

EXAMPLE 5. Let $M = \{a, b, \ldots, m\}$ be the set of the first thirteen letters, and let $\Delta = \{acd, bde, cef, dfg, egh, fha, gac, hcb, abi, cij, bjk, ikl, jlm, kma, lab, mbc, hcm, bhm\}$. Obviously, Δ is a PCO. Suppose that $\Delta^* \supset \Delta$ is a saturated PCO. If $abc \in \Delta^*$ then, since $acd \in \Delta^*$, also $bcd \in \Delta^*$. Then, also $cde \in \Delta^*$, and successive applications of transivity finally yield $acb \in \Delta^*$, which contra-

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