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EXISTENCE AND UNIQUENESS OF FUNCTIONAL CALCULUS HOMOMORPHISMS

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The purpose of this note is to announce a uniqueness result for the holomorphic functional calculus in commutative Banach algebras which is much stronger than the usual uniqueness assertion.

Let A be a commutative Banach algebra with unit and let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of elements of A. Denote by $\sigma(\mathbf{a})$ the joint spectrum of a and by $\mathcal{O}(\sigma(\mathbf{a}))$ the topological algebra of germs of functions holomorphic near $\sigma(\mathbf{a})$. The holomorphic functional calculus, developed by Shilov [6], Arens-Calderón [4] and Waelbroeck [7], provides a continuous unital homomorphism $\theta_{\mathbf{a}}$: $\mathcal{O}(\sigma(\mathbf{a})) \longrightarrow A$ such that:

(i) $\theta_{\mathbf{a}}(z_i) = a_i$ for $i = 1, 2, \ldots, n$,

(ii) $\theta_{\mathbf{a}}(f)\hat{} = f \circ (\hat{a}_1, \ldots, \hat{a}_n)$ for each f in $\mathcal{O}(\sigma(a))$,

where $\hat{b}$ denotes the Gelfand transform of an element b of A, acting on ΔA (the maximal ideal space of A). If $\mathbf{a}' = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m})$ is an $n + m$-tuple and π: $\mathbf{C}^{n+m} \longrightarrow \mathbf{C}^n$ is the projection, then the homomorphisms $\theta_{\mathbf{a}}$ and $\theta_{\mathbf{a}'}$ satisfy the following compatibility condition:

(iii) $\theta_{\mathbf{a}}(f) = \theta_{\mathbf{a}'}(f \circ \pi)$ for each f in $\mathcal{O}(\sigma(\mathbf{a}))$.

The usual uniqueness assertion is that the family $\{\theta_{\mathbf{a}}\}$ is unique subject to these requirements. However, we can show that the compatibility condition is redundant, and that the individual homomorphisms are themselves unique.

THEOREM 1. *The homomorphism $\theta_{\mathbf{a}}$ is the unique continuous unital homomorphism of $\mathcal{O}(\sigma(\mathbf{a}))$ into A which satisfies conditions* (i) *and* (ii) *above.*

Theorem 1 follows as an application of a more general existence and uniqueness result (Theorem 2, below). Let U be a domain in $\mathbf{C}^n$, $\widetilde{U}$ its envelope of holomorphy and $\mathcal{O}(U)$ the Fréchet algebra of holomorphic functions on U. Each holomorphic function f on U has a unique extension to $\widetilde{U}$, which we denote by $\widetilde{f}$. Note that each continuous unital homomorphism ϕ: $\mathcal{O}(U) \longrightarrow A$ with $\phi(z_i) = a_i$ for $i = 1, \ldots, n$ has a continuous adjoint $\phi^*: \Delta A \longrightarrow \widetilde{U}$ such that $\widetilde{z}_i \circ \phi^* = \hat{a}_i$ for $i = 1, \ldots, n$. The following result shows that this correspondence of ϕ with ϕ^* is bijective.

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