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THE ANTIPODE OF A FINITE-DIMENSIONAL HOPF ALGEBRA OVER A FIELD HAS FINITE ORDER

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The purpose of this note is to indicate a proof of the fact that the order of the antipode of a finite-dimensional Hopf algebra over a field has finite order. It has been shown [2], [5] that the order of the antipode of an infinite-dimensional Hopf algebra is not necessarily finite; and the order of the antipode is finite in the finite-dimensional case if the Hopf algebra is unimodular [1] or pointed and the ground field has prime characteristic [6]. Using the bilinear form introduced and studied in [3] we prove that the order is finite for any finite-dimensional Hopf algebra over a field.

The bilinear form, integrals, grouplike elements, one-dimensional ideals, and the antipode are all related in rather intriguing ways. The proof of the finite order theorem is based on an explicit formula describing the fourth power of the antipode, as suggested by Theorem 5.5 of [1]. Full details will appear elsewhere [4].

1. **The bilinear form.** Let A be a finite-dimensional Hopf algebra over a field k. Then the linear dual A^* is also a Hopf algebra. An element $x \in A$ is called a *left integral* if $hx = \epsilon(h)x$ for all $h \in A$. In [6] it is shown that the space of left integrals is one dimensional, and if $0 \neq m \in A^*$ is a left integral then $A^* = A \cdot m$ (hence A^* is a free left A-module). Let $0 \neq m \in A^*$ be a left integral. Then the associative bilinear form β: $A \times A \to k$ defined by $\beta(a, b) = m(ab)$ is nonsingular.

The bilinear form β lies at the foundation of the analysis of the antipode. Using it one can derive much information about bijective bialgebra maps as well (for example $t = s^2$ where s is the antipode of A). If t: $A \to A$ is any bijective bialgebra map then one can compute directly that $\beta(t(a), b) = \beta(a, \omega t^{-1}(b))$ for some $0 \neq \omega \in k$. Thus the transpose t^* of t with respect to the bilinear form β has a particularly nice characterization. Using the fact that $t^* = \omega t^{-1}$ it follows that the eigenvalues $\lambda_1, \ldots, \lambda_r$ of t are also $\lambda_1^{-1}\omega, \ldots, \lambda^{-1}\omega$ (in fact the relationship $\dim A_\lambda = \dim A_{\lambda^{-1}\omega}$ holds for eigenspaces, $\lambda \neq 0$). The invariant factors of t possess a high degree of symmetry dependent on ω.

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