BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 6, November 1975

ROTUNDITY, ORTHOGONALITY, AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

BY J. R. HOLUB[1]

Communicated by Alston Householder, May 30, 1975

1. The purpose of this paper is to announce a number of interesting new results concerning the geometry of normed linear spaces. In particular, we present some new characterizations of rotund normed linear spaces and of inner product spaces. The general theme connecting the two topics is the realization that special cases of conditions characterizing inner product spaces are often in themselves characterizations of rotundity.

Details and proofs will appear elsewhere.

Throughout the paper E will denote a real normed linear space (n.l.s.) and M a subspace of E, denoted $M \subset E$. If $\{x_i\}_{i=1}^n$ is a subset of E, $[x_i]_{i=1}^n$ will denote the linear span of $\{x_i\}$.

2. **Rotundity**. Recall that a n.l.s. E is said to be *rotund* [3] (or *strictly convex* [2]) if every point on the unit sphere in E is an extreme point. Our first result shows that rotundity is characterized by a very desirable condition involving the cone in E generated by a set of vectors.

DEFINITION 1. Let $\{x_i\}_{i=1}^2$ be a normalized linearly independent pair of vectors in E. Then $C\{x_i\} = \{a_1x_1 + a_2x_2 | a_1 \cdot a_2 \geqslant 0\}$ is called the *cone* of $\{x_i\}_{i=1}^2$ in E.

THEOREM 1. *E is rotund $\Leftrightarrow$ for any normalized linearly independent set $\{x_i\}_{i=1}^2$ in E, the set of points in $[x_i]_{i=1}^2$ equidistant from x_1 and x_2 is a subset of $C\{x_i\}$.*

Another characterization of rotundity which has a similar flavor is based on the following lemma which is interesting in its own right.

LEMMA 1. *Let E be a 2-dimensional n.l.s. Then every point $x \in E$ with $\|x\| < 1$ is the midpoint of a chord of the unit sphere in E (i.e. there exist x_1 and x_2 with $\|x_1\| = \|x_2\| = 1$ for which $x = (1 - \lambda)x_1 + \lambda x_2$, $0 < \lambda < 1$).*

A natural question concerns the uniqueness of such a chord (for $x \neq 0$). The answer is given by

---

AMS (MOS) subject classifications (1970). Primary 46B99, 46C05, 46C10.

[1]Research partially supported by NSF grant GP 33778.