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A DOUBLE SCALE OF WEIGHTED L^2 SPACES

BY REESE T. PROSSER[1]

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We assemble here for easy reference a number of related results on a certain family of weighted L^2 spaces over R^n. Most of these results are probably familiar to anyone who has thought about them at all, but they do not seem to be readily available in the literature. Proofs are to be submitted to Trans. Amer. Math. Soc.

Let S denote the space of tempered test functions (real or complex-valued) on R^n, and S' the dual space of tempered distributions [3]. Let $\hat{f}$ denote the Fourier transform of the function f. Now define on S the operators X and K by:

$$(Xf)(x) = (1 + x^2)^{1/2} f(x), \quad \widehat{(Kf)}(k) = (1 + k^2)^{1/2} \hat{f}(k)$$

and then define in terms of these operators on S the norms

$$\|f\|_{\alpha,\beta} = \|K^\alpha X^\beta f\|_2, \qquad -\infty < \alpha, \beta < +\infty,$$

$$\|f\|'_{\alpha,\beta} = \|X^\beta K^\alpha f\|_2, \qquad -\infty < \alpha, \beta < +\infty,$$

where $\|f\|_2$ is the usual L^2 norm of f.

Our first result relates these norms:

(1) For each α, β the norms $\| \ \|_{\alpha,\beta}$ and $\| \ \|'_{\alpha,\beta}$ are equivalent.

This follows directly from Leibnitz' rule if α is a positive even integer, but requires some form of interpolation theorem (cf. [2]) for other positive values, and a duality argument for negative values.

Now let $H(\alpha, \beta)$ denote the completion of S in the norm $\| \ \|_{\alpha,\beta}$. It is helpful to think of the family $H(\alpha, \beta)$ as a doubly-indexed scale of weighted L^2 spaces in which, roughly speaking, the second index describes the behavior of the function $f(x)$ as $|x| \longrightarrow \infty$, and the first the behavior of $\hat{f}(k)$ as $|k| \longrightarrow \infty$. It is often useful to consider these two modes of behavior separately, which

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