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SEMICONTINUITY OF KODAIRA DIMENSION

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Let X be a compact analytic space (or a complete algebraic variety) and let L be a line bundle on X and denote by $f_i\colon X \longrightarrow \mathbf{P}^N$ the rational map defined by the global sections of $L^{\otimes i}$. The L-dimension of X, $K(X, L)$ is defined by

$$K(X, L) = \overline{\lim_{i\to\infty}} (\dim(f_i(X))$$

with the convention $K(X, L) = -\infty$ if $L^{\otimes i}$ has no nontrivial sections for all $i > 0$. In the particular case when X is nonsingular and $L = \Omega$ is the canonical bundle, the invariant $K(X) = K(X, \Omega)$ is called the canonical (or Kodaira) dimension of X and is the fundamental invariant in the classification of surfaces. Recent works by Ueno [4] and Iitaka [1], [2] have studied $K(X, L)$ for higher dimensional varieties. A fundamental open question is the behavior of $K(X, L)$ under deformations of (X, L). When X is a smooth surface the plurigenera (and hence the Kodaira dimension) are deformation invariant [1], and Iitaka has constructed a family of threefolds X_t with $K(X_0) = 0$ and $K(X_t) = -\infty$, $t \neq 0$.

Our main result is

THEOREM. *Given X_0 a compact analytic space (or complete algebraic variety) and L_0 a line bundle on X_0 satisfying*

(1) *$L_0^{\otimes i}$ is spanned by its global sections for some $i > 0$,*

(2) *$K(X_0, L_0) = \dim(X_0)$,*

and (X_t, L_t) is any (flat) deformation of (X_0, L_0), then $K(X_t, L_t) = K(X_0, L_0)$.

When X_0 is a smooth surface and $L_0 = \Omega_0$ it was shown by Mumford [3] that hypothesis (1) on L_0 is implied by (2). For general L_0 hypothesis

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