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A UNIQUENESS THEOREM FOR HOMOLOGY IN $\mathcal{C}$at, THE CATEGORY OF SMALL CATEGORIES

BY DANA MAY LATCH

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I. **Introduction.** Oberst [7], Laudal [4], Watts [10], and André [1] have shown that derived functors of colimit define a homology theory for $\mathcal{C}$at, the category of small categories. In this note, we outline a proof of uniqueness for such a homology theory, making extensive use of a Kan-type construction (see e.g. Lemma A) and of uniqueness for homology in $\mathcal{S}^{\Delta^{\mathrm{op}}}$, the category of simplicial sets [2].

II. **Preliminaries.** The following Kan-type construction is used in several contexts.

LEMMA A. *Let $\mathcal{C}$ be a cocomplete category, $\mathbf{C}$ a small category, and θ: $\mathbf{C} \longrightarrow \mathcal{C}$ a functor. Then there exists an adjoint pair: the singular functor S_θ: $\mathcal{C} \longrightarrow \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}$ defined by $S_\theta(A) = \mathcal{C}(\theta\_, A)$, for $A \in |\mathcal{C}|$, and its left adjoint $\hat{\theta}$: $\mathcal{S}^{\mathbf{C}^{\mathrm{op}}} \longrightarrow \mathcal{C}$.*

Let Δ be the small category whose objects are the finite ordinals $[k] = \{0 < 1 < 2 < \ldots < k\}$ and whose morphisms are order preserving functions μ: $[k] \longrightarrow [m]$. By considering the full inclusion functor ι: $\Delta \longrightarrow \mathcal{C}$at, in the context of Lemma A, nerve, N: $\mathcal{C}$at $\longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$, is the singular adjoint of categorical realization c: $\mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{C}$at and $cN = \mathrm{id}_{\mathcal{C}\mathrm{at}}$ [3, p. 33]. Thus the standard representable k-dimensional simplicial set $\Delta[k]$ is actually $N([k]) = \Delta(-, [k])$.

Similarly, the functor τ: $\Delta \longrightarrow \mathcal{C}$at defined as the comma category, $\tau[k] = \Delta \downarrow [k]$, gives rise to another pair of adjoint functors S: $\mathcal{C}$at $\longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$ and Γ: $\mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{C}$at. Let $X \in |\mathcal{S}^{\Delta^{\mathrm{op}}}|$, then ΓX is the small category whose objects are $\amalg_{k \geqslant 0} X_k$, and whose morphisms are triples $\langle y, \mu, x \rangle$ where $x \in X_m$ is the codomain, μ: $[k] \longrightarrow [m]$ in Δ is the morphism, and $y = X(\mu)x$ in X_k is the domain.

The natural transformation "last", η: $\tau \stackrel{\cdot}{\longrightarrow} \iota$, is given by $\eta_k(\alpha$: $[p] \longrightarrow [k]) = \alpha(p) \in [k]$. By adjoint functor theory and by the theory of coends, η induces natural transformations $\eta^1: N \stackrel{\cdot}{\longrightarrow} S$, $\eta^2: \Gamma \stackrel{\cdot}{\longrightarrow} c$, $\eta^3: \Gamma N \stackrel{\cdot}{\longrightarrow} cNx = \mathrm{id}_{\mathcal{C}\mathrm{at}}$, and $\eta^4: N\Gamma \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathcal{S}^{\Delta^{\mathrm{op}}}}$.

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