BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 2, March 1975

STEENROD HOMOLOGY AND OPERATOR ALGEBRAS

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Communicated by I. M. Singer, September 25, 1974

The recent work of Larry Brown, R. G. Douglas, and Peter Fillmore (referred to as BDF) [2], [3], and [4] on operator algebras has created a new bridge between functional analysis and algebraic topology. This note and a subsequent paper [5] constitute an effort to make that bridge more concrete.

We first briefly describe the BDF framework. This requires the following C^*-algebras: $C(X)$, the continuous complex-valued functions on a compact metric space X; L, the bounded operators on an infinite dimensional separable Hilbert space; $K \subset L$, the compact operators; and L/K, the Calkin algebra. (Let π: $L \longrightarrow L/K$ be the projection.) An *extension* is a short exact sequence of C^*-algebras and C^*-algebra morphisms of the form $0 \longrightarrow K \longrightarrow E \longrightarrow C(X) \longrightarrow 0$ where E is a C^*-algebra containing K and I (the identity operator) and contained in L. Unitary equivalence classes of extensions form an abelian group, denoted $\mathrm{Ext}(X)$.

$\mathrm{Ext}(X)$ was invented by BDF in order to study essentially normal operators, that is, operators $T \in L$ with πT normal. Let E_T denote the C^*-algebra generated by I, T, and K, and let $X = \sigma(\pi T)$, the spectrum of πT. Then the exact sequence $0 \longrightarrow K \longrightarrow E_T \longrightarrow C(X) \longrightarrow 0$ represents an element of $\mathrm{Ext}(X)$. This element is zero if and only if T is a compact perturbation of a normal operator. For $X \subset \mathbf{C}$, BDF prove that

$$\mathrm{Ext}(X) \simeq \widetilde{H}^0(\mathbf{C} - X). \tag{1}$$

This isomorphism assigns to E_T a sequence of integers corresponding to the Fredholm index of $T - \lambda I$ on the various bounded components of $\mathbf{C} - X$.

The isomorphism (1) was subsequently generalized [3]. Let $E_{2n+1}(X) = \mathrm{Ext}(X)$ and $E_{2n}(X) = \mathrm{Ext}(SX)$, where SX is the suspension of X. Then BDF show that E_* satisfies (on compact metric pairs) all of the Eilenberg-Steenrod

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AMS (MOS) subject classifications (1970). Primary 46L05, 47B30, 55B15, 55B20, 55H25.

Key words and phrases. Brown-Douglas-Fillmore homology, generalized Steenrod homology, K-theory, Chern character.

[1] Research partially supported by NSF Grant GP29006.