

NECESSARY AND SUFFICIENT CONDITIONS FOR DETERMINING A HILL'S EQUATION FROM ITS SPECTRUM¹

BY WALLACE GOLDBERG²

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A Hill's equation is an equation of the form

$$(1) \quad y'' + [\lambda - q(z)]y = 0.$$

We assume $q(z + \pi) = q(z)$, where $q(z)$ is integrable over $[0, \pi]$. Without loss of generality, it is customary to assume that $\int_0^\pi q(z) dz = 0$. The discriminant of (1) is defined by $\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$ where y_1 and y_2 are solutions of (1) satisfying $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$.

The set of values of λ for which $|\Delta| > 2$ consists of a finite or an infinite number of finite disjoint intervals and one infinite interval. These intervals are called instability intervals, since (1) has no solution which is bounded for all real z in these intervals. When $|\Delta| < 2$, all solutions of (1) are bounded for all real z and the corresponding intervals are called stability intervals. Pertinent information about stability and instability intervals of (1) can be found in Magnus and Winkler [1].

In [2] it was proved that if $q(z)$ is real and integrable, and if precisely n finite instability intervals fail to vanish, then $q(z)$ must satisfy a differential equation of the form

$$(2) \quad q^{(2n)} + H(q, q', \dots, q^{(2n-2)}) = 0$$

where H is a polynomial of maximal degree $n + 2$. Explicit expressions of this equation are displayed in [2] and [3] for the cases $n = 0, 1$ and 2 .

For an infinite class of Korteweg-deVries equations of the form

$$q_t = K_n(q, q_z, \dots, \partial^{2n+1}q/\partial z^{2n+1}) \quad (n = 0, 1, 2, \dots),$$

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