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A LATTICE FIXED-POINT THEOREM WITH CONSTRAINTS[1]

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This paper presents a lattice fixed-point theorem having applications in game theory and elsewhere. The results presented here form part of the author's Ph.D. thesis in Operations Research, conducted under the supervision of Robert Wilson.

Let L be a complete lattice. Denote elements of L with small letters $a, b, c, \cdots$, and denote subsets of L with capital letters $A, B, C \cdots$. Consider a function $U: L \longrightarrow L$ with *property* P: for any $A \subseteq L$, $U(\bigvee A) = \bigwedge U(A)$, where $U(A) \equiv \{U(a) | a \in A\}$.[2] Denote the composition of U with itself by U^2.

Property P implies

LEMMA 1. *(1) $a \leqslant b$ implies $U(a) \geqslant U(b)$; (2) $a \leqslant b$ implies $U^2(a) \leqslant U^2(b)$.*

If we define $L_D(U) \equiv \{a \in L | a \leqslant U(a)\}$ and $L_D(U^2) \equiv \{a \in L | a \leqslant U^2(a)\}$, then we have

LEMMA 2. *(1) $U^2: L_D(U) \longrightarrow L_D(U)$; (2) $U^2: L_D(U^2) \longrightarrow L_D(U^2)$.*

LEMMA 3. *$L_D(U^2)$ is a complete join subsemilattice of L.*

We know from Tarski's theorem [1] that U^2 has a fixed point in L, while it is not generally true that U has a fixed point. We can, however, state the following

THEOREM. *There exists an element $s \in L$ such that $s = U^2(s)$ and $s \leqslant U(s)$.*

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2 Join and meet are represented by $\bigvee$ and $\bigwedge$.