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LIE ALGEBRA COHOMOLOGY OF CERTAIN INFINITE-DIMENSIONAL REPRESENTATIONS

BY M. S. OSBORNE

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Let G denote a connected semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$, and σ the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. The Lie algebra cohomology of certain nilpotent subalgebras of $\mathfrak{g}_c$ plays an important role in the Bott-Borel-Weil theorem and its generalizations, notably in the works of Kostant [**1**], Schmid [**3**], and others. An important consideration is the regular behavior of the group actions associated to the Lie algebra cohomology complex.

In this note we announce some results concerning the case where the nilpotent subalgebra mentioned above comes from an Iwasawa decomposition of $\mathfrak{g}$. This case is virtually the opposite of that considered by Schmid, and the aforementioned regularity is only partial.

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1. **The setting.** Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, θ the corresponding Cartan involution, $\mathfrak{j}$ a θ-stable Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{j}_c$ the complexification of $\mathfrak{j}$, $\mathfrak{b}_c$ a Borel subalgebra of $\mathfrak{g}_c$ containing $\mathfrak{j}_c$, and $\mathfrak{p}_c$ a parabolic subalgebra of $\mathfrak{g}_c$ containing $\mathfrak{b}_c$. Let $\mathfrak{n}_c^+$ denote the nil radical of $\mathfrak{p}_c$; $\mathfrak{n}_c^+$ will be used in computing cohomology. Let $\mathfrak{l}_c$ be a σ-stable Levi factor of $\mathfrak{p}_c$, and set $\mathfrak{l}=\mathfrak{l}_c\cap\mathfrak{g}$. Let L denote the centralizer in G of the center of $\mathfrak{l}$; L is then a reductive subgroup of G which normalizes $\mathfrak{n}_c^+$. In the work of Kostant, G is compact. In that of Schmid, $\mathfrak{p}_c=\mathfrak{b}_c$ and $\mathfrak{j}\subset\mathfrak{k}$. In the work of the author, $\mathfrak{p}_c$ is the complexification of a minimal parabolic subalgebra of $\mathfrak{g}$, so that $\mathfrak{n}_c^+$ is the complexification of the nilpotent constituent of an Iwasawa decomposition of $\mathfrak{g}$.

Let π be a representation of G in a Hilbert space $\mathscr{H}$, and assume that the Jordan-Hölder series of $\mathscr{H}$ is finite. Let $\mathscr{H}_\infty$ denote the space of C^∞-vectors in $\mathscr{H}$. One may form the Lie algebra cohomology complex $(C^q(\mathfrak{n}_c^+,\mathscr{H}_\infty), d^q)$ and its associated cohomology spaces $H^q(\mathfrak{n}_c^+,\mathscr{H}_\infty)$. Since $\mathscr{H}_\infty$ is a Fréchet space, so is $C^q(\mathfrak{n}_c^+,\mathscr{H}_\infty)$, and the differentials d^q are continuous linear maps. (n_c^+,π) will be called a CF-pair (for closed range, finite dimension) if each d^q has closed range and each $H^q(\mathfrak{n}_c^+,\mathscr{H}_\infty)$ is finite dimensional.

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