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A SEPARABLE SOMEWHAT REFLEXIVE BANACH SPACE WITH NONSEPARABLE DUAL

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ABSTRACT. An example is given of a separable Banach space X whose dual is not separable, but each infinite-dimensional subspace of X contains an infinite-dimensional subspace isomorphic to Hilbert space. Thus X contains no subspace isomorphic to c_0 or l_1, X is somewhat reflexive, and no nonreflexive subspace has an unconditional basis.

It has been conjectured that every infinite-dimensional Banach space has an infinite-dimensional subspace that is either reflexive or isomorphic to c_0 or to l_1 [**9**, p. 165]. A counterexample would also be an example of a space that has no infinite-dimensional subspace with an unconditional basis [**6**, Theorem 2, p. 521]. It is known that there is a nonreflexive Banach space J with no subspace isomorphic to c_0 or to l_1 [**6**, pp. 523–527], but J^{**} is separable. Each of the following is a necessary and sufficient condition for a separable Banach space X to contain a subspace isomorphic to l_1; separability is not needed for conditions (i) and (ii) (see [**5**, Theorem 2.1, p. 13] and [**10**, p. 475]).

(i) $L_1[0, 1]$ is isomorphic to a subspace of X^*.

(ii) $C[0, 1]^*$ is isomorphic to a subspace of X^*.

(iii) X^* has a subspace isomorphic to $l_1(\Gamma)$ for some uncountable Γ.

A natural and well-known conjecture in view of the preceding is that a Banach space has a subspace isomorphic to l_1 if the space is separable and its dual is not separable (e.g., see [**1**, §9, p. 243], [**2**, §5.4, p. 174], and the last paragraph of [**11**]). It will be shown that this conjecture is false. The counterexample $\tilde{X}$ has the property that each infinite-dimensional subspace has an infinite-dimensional subspace isomorphic to Hilbert space. Thus $\tilde{X}$ is also a counterexample to the conjecture that each separable somewhat-reflexive space has a separable dual (see [**3**, Problem 3, p. 191] and [**8**, Remark IV.2, p. 86]). Also, neither c_0 nor l_1 has an infinite-dimensional subspace isomorphic to Hilbert space, so no nonreflexive subspace has an unconditional basis [**6**, Theorem 2, p. 521]. It has been

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