BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

SETS OF COLORINGS OF CIRCUITS

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Communicated by Gian-Carlo Rota, November 6, 1973

1. **Introduction.** A circuit Γ is a triangulation of the one-dimensional sphere S^1. It shall have as its set of vertices $\Gamma_0 = Z_k = \{0, 1, \cdots, k-1\}$, and as its set of one-simplices $\Gamma_1 = \{\sigma_j = (j-1, j) | j = 1, 2, \cdots, k\}$. A coloring of Γ is a zero-dimensional cochain $c^0 \in C^0(\Gamma, Z_2 \oplus Z_2)$ whose coboundary is "nowhere zero", i.e. $\delta c^0(\sigma_j) \neq 0$ for all $\sigma_j \in \Gamma_1$. A set K of colorings of Γ is realizable as a set of admissible colorings if there is a triangulated two-dimensional disk D with boundary Γ such that the restriction homomorphism

$$j^{\#}: C^0(D, Z_2 \oplus Z_2) \to C^0(\Gamma, Z_2 \oplus Z_2)$$

(induced by the inclusion $j: \Gamma \to D$) takes the colorings of D onto K.

Let $\psi(k)$ be the minimum cardinality of a set K which is realizable as a set of admissible colorings.

REMARK 1. $\psi(k) = 0$ if and only if the four color conjecture is false.

The conjecture of Albertson and Wilf [**1**]. $\psi(k) = 3 \cdot 2^k$ for $k = 3, 4, \cdots$.

Comment 1. Since $3 \cdot 2^k$ is the number of colorings of any disk D with no interior vertices and k vertices in $\Gamma_0 = D_0$, we conclude $3 \cdot 2^k \geqq \psi(k)$.

Comment 2. It is not known whether the four color conjecture implies the Albertson-Wilf conjecture for $k > 6$. (It does for $k = 3, 4, 5$ and 6 [**1**].)

In [**1**], Albertson and Wilf announce:

THEOREM 1. *If the four color conjecture holds then*

$$\psi(k) \geqq (4!)F_{k-1} \geqq C((1 + \sqrt{5})/2)^k$$

where F_k is the kth Fibonacci number.

By generalizing the notion of a set of admissible colorings of Γ to the notion of a complete set of colorings of Γ, one can prove by induction on k:

THEOREM 2. *If the four color conjecture holds then*

$$\begin{aligned} \psi(k) &> 4 \cdot 3^{k/2} \quad \text{if } k \text{ is even,} \\ &> 8 \cdot 3^{(k-1)/2} \quad \text{if } k \text{ is odd.} \end{aligned}$$

AMS (MOS) subject classifications (1970). Primary 05C15.

Key words and phrases. Four color problem, set of all colorings.