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A REMARK ON THE LINDELÖF HYPOTHESIS

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I. **Introduction.** In this note, we sketch a development which offers new insight into some previous work on the Lindelöf hypothesis (LH).

As is well known [**5**, p. 276], the following two statements are equivalent to the LH:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\,dt = O(T^{1+\varepsilon}) \quad \text{for each } \varepsilon > 0 \text{ and } k \geqq 1; \tag{1}$$

$$\int_0^\infty |\zeta(\tfrac{1}{2} + it)|^{2k} e^{-\delta t}\,dt = O(\delta^{-1-\varepsilon}) \quad \text{for each } \varepsilon > 0 \text{ and } k \geqq 1. \tag{2}$$

That (1)$\Leftrightarrow$(2) follows from an elementary Tauberian argument. At present, (1) and (2) are known only for $k=1$, $k=2$.

According to the general formalism of Titchmarsh [**5**, pp. 137–138],

$$\int_0^\infty |\zeta(\tfrac{1}{2} + it)|^{2k} e^{-2\delta t}\,dt = O(1) + \int_0^\infty |\phi_k(ixe^{-i\delta})|^2\,dx,$$

where $\phi_k(z) = \sum_{n=1}^\infty d_k(n)e^{-nz}$ + residue term. Hopefully, one could expand the ϕ_k integral and estimate the resulting infinite series. This does not seem feasible, however, unless $\phi_k(z)$ satisfies a certain approximate functional equation (AFE); see [**5**, p. 147] and [**6**, p. 42]. This is one reason why only $k=1$, $k=2$ are known.

However, Bellman [**2**] has shown that, if the e^{-nz} in $\phi_k(z)$ are replaced by so-called Voronoi functions, one will always get an AFE. Unfortunately, these Voronoi functions have proved too messy to be useful computationally.

It would therefore be of interest to see what could be done with a method which involves much simpler functions.

II. **Development of the main theorem.** We base our development on the series

$$\sum_{n=1}^\infty n^{Q/A}\, d_k(n) \exp(-zn^{1/A})$$

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