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SQUARE-INTEGRABLE KERNELS AND GROWTH ESTIMATES FOR THEIR SINGULAR VALUES

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Let $K(x, y)$, $0 \leqq x, y \leqq \pi$, be Lebesgue square-integrable. Define

$$K^{(r)}(x, y) \equiv \partial^r K(x, y)/\partial x^r \qquad (r = 0, 1, 2, \cdots, s)$$

for nonnegative integer s, and assume that $K(x, y)$ is extended, as an even function of x if s is even, and as an odd function of x if s is odd, into the domain $-\pi \leqq x \leqq 0$, and thence as a periodic function of x with period 2π. Let the singular values μ_n, where

$$\phi_n(x) = \mu_n \int_0^\pi K(x, y)\Psi_n(y)\, dy,$$

$$\Psi_n(x) = \mu_n \int_0^\pi \overline{K(y, x)}\phi_n(y)\, dy$$

with $\|\phi_n\|$, $\|\Psi_n\| \neq 0$, be ordered (indexed) in the natural manner according to increasing size, namely $0 < \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots$.

In a perhaps overlooked paper, Smithies [**8**] has shown that

THEOREM 1. *If the $K^{(r)}(x, y)$, $0 \leqq r \leqq s-1$, are continuous in x, a.e. in y, and $K^{(s)}(x, y)$ is in $\mathscr{L}^p(x)$, a.e. in y, for some $1 < p \leqq 2$, then*

$$(1) \qquad \int_0^\pi \left[\int_0^\pi |K^{(s)}(x + h, y) - K^{(s)}(x - h, y)|^p\, dx \right]^{2/p} dy \leqq C\, |h|^{2\alpha}$$

for constant C, where $\alpha > 0$ if $s > 0$, $\alpha > (2-p)/2p$ if $s = 0$, implies

$$1/\mu_n = O(1)/n^{\alpha+s+1-1/p} \qquad \text{as } n \to \infty.$$

Recognizing (1) as essentially an integrated Lipschitz condition, and using various properties associated with the class of kernels which satisfy such a condition, we can substantially generalize the above result.

We say that $K^{(s)}(x, y)$ is in Lip α if

$$|K^{(s)}(x + h, y) - K^{(s)}(x - h, y)| < |h|^\alpha A(y) \qquad (0 < \alpha \leqq 1)$$

where $A(y)$ is nonnegative and square-integrable. Likewise $K^{(s)}(x, y)$ is said to be (relatively uniformly) of bounded variation if for all $N \geqq 1$ and

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