BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, March 1974

ONE-SIDED APPROXIMATION AND VARIATIONAL INEQUALITIES

BY UMBERTO MOSCO AND GILBERT STRANG[1]

Communicated by Hans Weinberger, May 1, 1973

ABSTRACT. For piecewise linear approximation of the unilateral Laplace equation (also known as the *obstacle problem*, and governed by a variational inequality), we prove that the gradient of the error $u-u_h$ is of order h. The proof rests on approximation of nonnegative functions U by nonnegative splines $V_h \leqq U$.

We are interested in one of the first and most fundamental of the variational problems introduced by Fichera, Stampacchia, and Lions [3], [4], [6]:

Find that function u in the convex set

$$K = \{v \mid v \in \mathscr{H}_0^1(\Omega),\ v \geqq \psi \text{ on } \Omega\}$$

which minimizes

$$I(v) = a(v, v) - 2(f, v) = \iint_\Omega (v_x^2 + v_y^2 - 2fv)\, dx\, dy.$$

If the "obstacle function" ψ were absent, this would be the classical Dirichlet problem for Poisson's equation $-\Delta u = f$, and the condition for a minimum would be a variational equation: $a(u, v) = (f, v)$ for v in $\mathscr{H}_0^1$. This is the weak form of Poisson's equation, and coincides with the engineer's "equation of virtual work".

For minimization over K instead of the full space $\mathscr{H}_0^1$, the variational equation turns into an inequality—just as, for minimization of a function g over $0 \leqq x \leqq 1$, the possibility of minima at the endpoints alters the usual $dg/dx = 0$. The condition that u be minimizing is

$$(1) \qquad a(u, v-u) \geqq (f, v-u) \quad \text{for all } v \text{ in } K.$$

Suppose we solve this problem approximately, by the Ritz principle: *The approximation u_h minimizes the functional I over a finite-dimensional*

AMS (MOS) subject classifications (1970). Primary 35J20, 65N30, 41A15.

Key words and phrases. Approximation, splines, variational problem, Ritz method.

[1] The first author was supported by GNAFA-CNR and enjoyed the hospitality of the Courant Institute under a CNR-NATO grant. The second author was supported by the National Science Foundation (P22928).