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QUASI-KAN EXTENSIONS FOR 2-CATEGORIES

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1. **Introduction.** Let $\mathscr{C}at$ denote the category of small categories and functors. $\mathscr{C}at$ is a Cartesian closed category, [2] and the prefix 2- will denote categories and functors enriched in $\mathscr{C}at$. 2-$\mathscr{C}at$ denotes the category of small 2-categories and 2-functors. It is also Cartesian closed, but there is another notion of a transformation between 2-functors F and G which has interesting properties; namely a *quasi-natural* transformation from F to G is a family of morphisms $\{\varphi_A : F(A) \to G(A)\}$ together with a family of 2-cells $\{\varphi_f : G(f)\varphi_A \to \varphi_B F(f)\}$ as illustrated

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\varphi_f"] & F(B) \arrow[d, "\varphi_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

satisfying obvious compatibility conditions. (The case where the φ_f's are isomorphisms has been considered in [7] and [8], but we make no such restriction.) Given this notion of "natural transformation", it is reasonable and useful to inquire about the corresponding notion of "quasi-limit" or, more generally, "quasi-Kan extension".

Such a Kan extension was used in an essential way for the proof of the main result in [4, §9], but until now no justification has been given for calling the construction used there a "Kan extension". In the usual case, if $S : \mathscr{A} \to \mathscr{B}$ is an ordinary functor and $\mathscr{X}$ is a cocomplete category, then under appropriate hypotheses the functor

$$\mathscr{X}^S : \mathscr{X}^B \to \mathscr{X}^A$$

is right adjoint to the (left) Kan extension $\Sigma S : \mathscr{X}^A \to \mathscr{X}^B$. ΣS can be constructed as follows: replace S by its associated factorization through an opfibration

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "Q_S"] & (S, \mathscr{B}) \arrow[l, shift left, "P"] \arrow[r, "P_S"] & \mathscr{B} \end{tikzcd}$$

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