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PROPERTIES OF RANK PRESERVING WEAK MAPS

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Let P_X be the set of all combinatorial pregeometries (or matroids) [5] on the n-set X. P_X is partially ordered by $G \rightarrow H$ (in lieu of $G \geqq H$) whenever the identity mapping from G to H is weak (i.e. each independent set of H is independent in G). A map which preserves geometric rank will be written $G \xrightarrow{*} H$. A map $G \rightarrow H$ is called *simple* if G covers H in the partial order on P_X.

Any map, $G \rightarrow H$, has a *decomposition* into simple maps (just select a saturated chain between G and H). In general such a decomposition will consist of some simple maps which are rank preserving and others which reduce rank (see Figure 1). Rank reducing simple maps are always rank one upper truncations (in the language of Crapo [4], a truncation, $G \rightarrow H$, is simple iff G is a co-atom in the lattice of erections of H). A result of Kennedy [6] shows that the truncation of any erectable pregeometry is never simple. Thus any weak map is decomposed into an alternating sequence of upper truncations and rank preserving weak maps. Upper truncations and their properties are well understood. The object of this research is to study the properties of rank preserving weak maps.

The following lemma collects some preliminary facts which are easily proved by checking independent sets.

LEMMA 1. *For G and H in P_X, $G \xrightarrow{*} H$, and p in X, if G/p denotes the contraction of G by p, $G-p$ the deletion of p from G, and $\tilde{G}$ the Whitney dual of G ([1] or [5]), then*

(a) *$G/p \xrightarrow{*} H/p$ provided p is not a loop of H.*
(b) *$G-p \xrightarrow{*} H-p$ provided p is not an isthmus of H.*
(c) *$\tilde{G} \xrightarrow{*} \tilde{H}$.*

Any minor $M=(G/A')-B'$ of G can be expressed as $M=(G/A)-B$ in which no loop is contracted and no isthmus deleted. In Proposition 1 and Corollaries 1 and 6 we assume that the indicated minors are expressed in this manner.

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