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SMOOTH S^1 ACTIONS AND BILINEAR FORMS[1]

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Let S^1 denote the multiplicative group of complex numbers of norm 1. Let X denote a smooth S^1 manifold, i.e., X consists of an underlying smooth manifold denoted by $|X|$ together with a smooth action of S^1. The equivariant complex K theory of X is $K_{S^1}^*(X) = K_{S^1}^0(X) \oplus K_{S^1}^1(X)$. It is a module over $R(S^1)$ the complex representation ring of S^1. This is the ring $Z[t, t^{-1}]$. For our purposes there are two important sets of prime ideals in $Z[t, t^{-1}]$:

(i) the set P_1 consisting of the principal ideals of the form $\mathfrak{p} = (\Phi_{p^r}(t))$ generated by the cyclotomic polynomial $\Phi_{p^r}(t)$ associated to the prime power p^r, i.e., $P_1 = \{(\Phi_{p^r}(t)) \mid \forall \text{ primes } p \text{ and integers } r\}$.

(ii) the set $P = \{(\Phi_m(t)) \mid \forall \text{ positive integers } m\}$.

The localized ring $R(S^1)_P$ is denoted by R. It is the subring of the field of fractions of $R(S^1)$ consisting of fractions a/b with b prime to all the ideals of P. Let $K_{S^1}^*(X)_P = K_{S^1}^*(X) \otimes_{R(S^1)} R$. The Atiyah-Singer index homomorphism [1] $\mathrm{Id}_{S^1}^X : K_{S^1}^0(TX) \to R(S^1)$ induces a homomorphism

$$\mathrm{Id}^X : K_{S^1}^0(TX)_P \to R.$$

Here TX is the tangent bundle of X and $|X|$ is compact without boundary. Suppose that $|X|$ is a spin^c manifold. Then there is an isomorphism

$$K_{S^1}^*(X)_P \xrightarrow{\Delta^X} K_{S^1}^*(TX)_P$$

of R modules [6] and we can define an R valued bilinear form $\langle\ \rangle_X$ on $K_{S^1}(X)_p$ by

$$\langle a, b\rangle_X = \mathrm{Id}^X(\Delta^X(a) \cdot b).$$

THEOREM 1 [2]. *The bilinear form $\langle\ \rangle_X$ is nonsingular, i.e., the associated homomorphism*

$$K_{S^1}^*(X)_P \xrightarrow{\Phi^X} \mathrm{Hom}_R(K_{S^1}^*(X)_P, R)$$

is surjective where $\Phi^X(a)[b] = \langle a, b\rangle_X$.

This result was conjectured in a similar form in [6].

A useful consequence of Theorem 1 is this: Set $K_{S^1}^*(X) = K_{S^1}^*(X)_P/T_X$ where T_X denotes the R torsion subgroup of $K_{S^1}^*(X)_P$. The bilinear form

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