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THE COMPREHENSIVE FACTORIZATION OF A FUNCTOR

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In this article we show that every functor has a factorization into an initial functor followed by a discrete 0-fibration and that this factorization is functorial. Size considerations will be ignored but may be easily filled in; we assume the existence of a category of sets large enough to dwarf any given finite number of categories.

There is an analogy between the category **Set** of sets and the category **Cat** of categories which is partly explained by the observation that each is a category of types for a suitable hyperdoctrine. A hyperdoctrine (Lawvere [**4**]) consists of a category T of types and a functor $P: T^{\mathrm{op}} \to \mathbf{Cat}$ satisfying conditions. The comprehension schema (also see [**4**]) is expressed by a pair of adjoint functors

$$PX \leftrightarrows T/X$$

for each object X of T, where T/X is the category of objects over X. It is often the case that this structure arises from more usual structure on the category T; namely:

(1) a factorization system (E, M) on T;

(2) a category object Ω in T which "classifies the M-subobjects".

The sense in which (1) is intended is that of Freyd-Kelly [**2**]. We say that Ω classifies the M-subobjects when there is a natural equivalence of categories $T(X, \Omega) \approx M(X)$, where $M(X)$ is the full subcategory of T/X consisting of the arrows in M with target X. Then $PX = T(X, \Omega)$. From (1), the functor "take the (E, M)-image" is the left adjoint of the inclusion $M(X) \to T/X$; this adjunction combines with (2) to yield the comprehension schema.

The familiar example of a hyperdoctrine which arises in the above way is provided by the power-set functor $P: \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$. Here M consists of monomorphisms, E of epimorphisms, and Ω is a set with two elements. These considerations lie at the heart of the elementary theory of the category of sets in the new elegant form—elementary topos theory—due to Lawvere-Tierney (see Freyd [**1**]). A topos T is a finitely complete, cartesian closed category satisfying (2) where M consists of the mono-

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